Finishing Graphs of polynomials (rough sketches)

In understanding these graphs we are to know the end-behavior and crossing/touching of x-intercepts.

Remember that the **Lead Coefficient Test** determines end-behavior

1. Even degree positive lead coefficient is UP on left and UP on right (i.e. UP/UP)
2. Even degree negative lead coefficient is DOWN/DOWN
3. Odd degree positive lead coefficient is DOWN/UP
4. Odd degree negative lead coefficient is UP/DOWN

i.e. Even degree polynomials behave roughly like \( y = a \ x^2 \) and odd degree behave roughly like \( y = a \ x^3 \).

Graphs of polynomials are **smooth** (no breaks or corners) and if the degree is \( n \)

1. the number of x-intercepts is at most \( n \)
2. the number of turning points is at most \( n-1 \)

When we are sketching or identifying a sketch we must get the end-behavior correct, and if the x-intercepts are known, the crossing versus touching behavior of the x-intercepts.

If \( (x-c) \) is a factor of the polynomial \( f(x) \), and if we can even get \( (x-c)^k \) as a factor but no bigger power of \( (x-c) \), then \( k \) is called the **multiplicity of the zero or x-intercept** \( c \)

1. If an x-intercept \( x=c \) has even multiplicity, then the graph **touches** at \( x=c \)
2. If an x-intercept \( x=c \) has odd multiplicity, then the graph **crosses** at \( x=c \)

Sketch some graphs:

#26 from book: \( f(x)=3(x+5)(x+2)^2 \) we can see that the lead coefficient is 3 and the degree is also 3 (add up the multiplicities of those factors). We can see that \( x=-5 \) is an x-intercept of multiplicity 1, \( x=-2 \) is an x-intercept of multiplicity 2.

**THUS** end behavior is DOWN/UP, graph **crosses** at \( x=-5 \), graph **.touches** at \( x=-2 \)

Additionally we can determine the y-intercept by subbing in \( x=0 \), we get \( 3 \times 5 \times 4 = 60 \)

Below is an accurate sketch of the graph.
we can see right now that the degree is 3 and the lead coefficient is -1, so end-behavior is UP/DOWN to get the x-intercepts we have to factor. More about finding x-intercepts later, but with this one, it turns out to factor easily by grouping:

\[-x^2(x+1)+4(x+1) = (-x^2+4)(x+1) = -(x^2-1)(x+1)\]

so finally it fully factors as

\[f(x) = -(x-2)(x+2)(x+1)\]

thus, we have x-intercepts: x=2; x=-2; x=-1 and all have multiplicity equal 1, so we have crossing at each x-intercept. (also remember there are no other x-intercepts!)

Summary: graph starts out way up high on the left, comes down, crosses at x=-2, quickly turns around and crosses again at x=-1, we can see that the y-intercept is 4 (from original equation), and finally the graph turns around again and crosses at x=2 and continues DOWN on the right end.

Here's the graph:
Now we play the reverse game. We see a graph and have to determine the function by identifying the factors (equal x-intercepts) and the lead coefficient (the sign of lead coefficient from lead coefficient test, and if given an additional graph value, by subbing in x, y as we did with polynomial questions that I called “find a”).

Here’s our first graph:

Our analysis: by lead-coefficient test: even degree with positive lead because it is UP/UP

crosses at x=-2, touches at x=0, crosses at x=1 thus the function has the form:
\[
f(x) = a(x + 2)x^2(x - 1)
\]

and we know that \( a > 0 \). To find the actual value of \( a \) suppose we are told that \( f(-1) = -4 \).

then we sub in \( x = -1 \) and \( y = -4 \):

\[
-4 = a(-1 + 2)(-1)^2(-1 - 1) = a(1)(1)(-2)
\]

and we deduce that the value of \( a \) must be 2 (remember we knew that it was positive).

So our final answer is \( f(x) = 2(x + 2)x^2(x - 1) \).

Another example: even without a picture we are told that the graph is UP/DOWN that it touches at \( x = 4 \) and crosses at \( x = -3 \) and has \( y \)-intercept 5. Determine exactly what \( f(x) \) is.

EASY!! \( f(x) \) has odd degree with a negative lead coefficient. \((x + 3)^2\) is a factor, and so is \((x - 4)\) so we have so far that \( f(x) = a(x + 3)(x - 4)^2 \) and to determine \( a \) we sub \( x = 0 \) and \( y = 5 \) because we have been told that \((0, 5)\) is a point on the graph. Remember as a double check, we know \( a < 0 \).

So \( 5 = a(0 + 3)^2(0 - 4) \) and determine that \( a = -5/36 \) so our final answer is

\[
f(x) = -\frac{5}{36}(x + 3)^3(x - 4)
\]

Here's the actual graph:
Section 3.3 and 3.4

More factoring and x-intercept tips

Factor Theorem: If \( f(x) \) is a polynomial and \( c \) is any number, then

\[ x-c \text{ is a factor of } f(x) \text{ if and only if } f(c) = 0 \]

i.e. Factor/ zero / root / x-intercept are all the same thing.

Generally if we can find one zero of \( f(x) \) we can “factor it out” (we learn how in a minute) and that will give us a quotient (the part of \( f(x) \) left after factoring out \( x-c \)) and we could “keep going”.

A helpful tool (to memorize) is

The Rational Zero Theorem: (this is a list of all possible fractions that might be zeros)

The possible rational zeros of \( f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 \) are

all the ones in the list \( \pm \frac{\text{factors of } a_0 (\text{constant term})}{\text{factors of } a_n (\text{lead term})} \) (all possible combinations)

Example: What are the possible rational solutions of \( 5x^6 - 4x^3 + 17x + 14 = 0 \) ??

The only thing that matters here are the lead coefficient 5 and the constant term 14

the factors of 5 are 1,5 and the factors of 14 are 1,2,7,14 so the list is

\[ \pm \frac{1,2,7,14}{1,5} = \pm 1, \pm \frac{1}{5}, \pm 2, \pm \frac{2}{5}, \pm 7, \pm \frac{7}{5}, \pm 14, \pm \frac{14}{5} \]

That's it. That list is the answer.

Example: Could 4 be a zero of \( f(x) = 8x^3 - 19x^2 + 803x - 26 \) ?

Solution: Use rational zero theorem to make the list of possible rational zeros:

factors of 26 (i.e. 1, 2, 13, 26) go in the numerator, and the factors of 8 (i.e. 1,2,4,8) go in the denominator.

The number 4 is NOT of the form +/- (factor of 26) over (factor of 8),

So ANSWER: NO 4 is not a possible zero of \( f(x) \).
Dividing polynomials: LONG or SYNTHETIC

If \( f(x) \) is a polynomial and we are wondering if \( x-c \) is a factor we can divide to get

\[
\frac{f(x)}{x-c} = q(x) + \frac{r}{x-c}
\]

\( r \) is remainder, \( q(x) \) is quotient

**Remainder Theorem:** The remainder \( r \) is ALWAYS the value of \( f(c) \) when we are dividing \( f(x) \) by \( x-c \). (remember that if \( f(c) = 0 \) then \( x-c \) is a factor and the remainder is 0 by Factor Theorem)

**Example:** to show synthetic division. Find \( q(x) \) and \( r \) when \( \frac{5x^2-18x-8}{x-4} \) (4 is a *good* guess for a zero because of the rational zero theorem)
To synthetically divide by \(x-4\) we use the value 4 in first position in this table format:

<table>
<thead>
<tr>
<th>4</th>
<th>5</th>
<th>-18</th>
<th>-8</th>
</tr>
</thead>
</table>

\[4 \mid 5 -18 -8 \]

\[/ \]

\[<<<\] coefficients of \(f(x)\) with 0's inserted for any missing powers

\[<<<\] leave a line space for our work

\[--------------------
5
\]

\[<<<\] quotient and remainder will be here

\[\]

\[\]

start by simply bringing the first number down. Next compute \(4 \times 5 = 20\) and see where it goes

<table>
<thead>
<tr>
<th>4</th>
<th>5</th>
<th>-18</th>
<th>-8</th>
</tr>
</thead>
</table>

\[4 \mid 5 -18 -8 \]

\[/ \]

\[20 \]

\[<<<\] the quantity 4 times the last number entered (5) goes here

\[--------------------
5 \]

\[2 \]

\[<<<\] that number 2 comes from simply adding

we repeat by computing 4 times 2 which is 8

<table>
<thead>
<tr>
<th>4</th>
<th>5</th>
<th>-18</th>
<th>-8</th>
</tr>
</thead>
</table>

\[4 \mid 5 -18 -8 \]

\[20 \]

\[8 \]

\[--------------------
5 \]

\[2 \]

\[0 \]

\[<<<\] this last position is the remainder, We got remainder 0

\[\]

\[\]

these guys are the coefficients of \(q(x)\), hence \(q(x) = 5x + 2\) and \(r = 0\)

we have shown that \(\frac{5x^2 - 18x - 8}{x-4} = 5x + 2 + \frac{0}{x-4}\) but remainder 0 can be ignored, so we have actually shown that \(x-4\) is a factor and that \((\text{by multiplying both sides by } x-4)\)

\[5x^2 - 18x - 8 = (5x + 2)(x-4)\]

and this process works no matter the degree of \(f(x)\).

**Example #28 from book:** Find quotient \(q(x)\) and remainder \(\frac{x^7 + x^5 - 10x^3 + 12}{x+2}\) (if remainder is 0, conclude that \(f(x) = q(x)(x+2)\); while if remainder \(r\) is not 0, remember this is value of \(f(-2))\).

Working from left to right, I'll show all steps of synthetic division by -2 (i.e. \(x+2\))

<table>
<thead>
<tr>
<th>-2</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>-10</th>
<th>0</th>
<th>12</th>
</tr>
</thead>
</table>

\[\]

\[<<<\] had to fill in the missing powers

<table>
<thead>
<tr>
<th>-2</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>-10</th>
<th>20</th>
<th>-20</th>
<th>40</th>
<th>-80</th>
</tr>
</thead>
</table>

\[\]

\[<<<\] remainder is -68, hence \(f(-2) = -68\) NOT a zero.

\[\]

\[\]

coefficients of \(q(x)\), thus
Example to USE the rational zero theorem to fully solve (factor) a polynomial.

#44: Solve \( 2x^3 - 3x^2 - 11x + 6 = 0 \) Procedure: use the rational zero theorem to make good guesses for ONE of the solutions (or sometimes the first guess is made for us). Then use synthetic division to factor it out (finding the remainder and quotient)

The possible rational zeros are \( +/- \{ \text{factors of 6} \} \) over \( \{ \text{factors of 2} \} \); i.e. \( \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{2}{2}, \pm \frac{3}{2}, \pm \frac{6}{2} \)

the book informs us that \( x = -2 \) is a zero (but if we weren't told this, we could just start testing values from this list). Hint: we can also test by graphing calculator (I like the TABLE feature). Once we find a zero, we factor it out with synthetic division, then keep working on the quotient \( q(x) \).

Divide out \( x+2 \), i.e. Use \( -2 \)

\[
\begin{array}{c|cccc}
  & 2 & -3 & -11 & 6 \\
  \hline
-2 & & & & \\
  \hline
 & -4 & 14 & -6 \\
  \hline
 & 2 & -7 & 3 & 0 \\
\end{array}
\]

\( \text{<<< yep, book was right, remainder is 0 so} x+2 \text{ is a factor.} \)

\( q(x) \) is \( 2x^2 - 7x + 3 \) so now we have \( (2x^2 - 7x + 3)(x+2) \)

Of course once you have a quadratic factor, we have easy ways to find all zeros of it. But let's continue with this method. The possible rational zeros of this quadratic are \( +/- \{1,3\} \) over \( \{1,2\} \)

Caution: an equation can have zeros that are not rational zeros; e.g. Quadratic formula

We are factoring \( q(x) = 2x^2 - 7x + 3 \); let's try subbing in \( x=1 \), value is \( q(1) = -2 \); \( q(-1) = 12 \), so they don't work. Let us try \( 3 \) to see if \( x-3 \) is a factor

\[
\begin{array}{c|cccc}
  & 2 & -7 & 3 \\
  \hline
3 & & & & \\
  \hline
 & 6 & -3 \\
  \hline
 & 2 & -1 & 0 \\
\end{array}
\]

\( \text{<<< remainder is 0, and new quotient is } 2x - 1 \text{ (which is 0 when } x = 1/2) \)
SO we have found all solutions to \( 2x^3 - 3x^2 - 11x + 6 = 0 \) (1/2, 3, and -2)

and have fully factored \( 2x^2 - 7x + 3 \) to be \((2x-1)(x-3)\)

and \( 2x^3 - 3x^2 - 11x + 6 = (2x-1)(x-3)(x+2) \)
Hand-in

1. Find a polynomial \( f(x) \) that might have the graph below (in this case, just use lead coefficient to be either 1 or -1 as appropriate)

2. List the possible rational zeros of \( 3x^3 + 35x^2 + 87x - 33 \)

3. Use synthetic division to find the quotient and remainder if we are testing if \( x = \pm \frac{1}{3} \) is a zero of \( 3x^3 + 35x^2 + 87x - 33 \)