ON RANČIN’S PROBLEM

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Abstract. Few observations on a paper of Arhangel’skii and Buzyakova led us to consider Rančin’s problem. The main result here is the construction under ◦ of a compact c-sequential space that is not sequential.

1. Hušek number and c-sequentiality

All spaces are assumed $T_2$. For undefined notions we refer to [6]. Given a space $X$ and a point $x ∈ X$, the Hušek number $\text{Hus}(x, X)$ is the smallest cardinal $κ$ such that for any set $A ⊆ X \setminus \{x\}$ of regular cardinality $|A| ≥ κ$ there exists an open neighbourhood $U$ of $x$ such that $|A| = |A \setminus U|$ [1]. Clearly, we always have $\text{Hus}(x, X) ≤ ψ(x, X)^+$. As is standard, $\text{Hus}(X) = \sup\{\text{Hus}(x, X) : x ∈ X\}$.

A space is linearly Lindelöf if every open cover which is totally ordered by inclusion has a countable subcover. Equivalently, $X$ is linearly Lindelöf if every subset of uncountable regular cardinality has a complete accumulation point.

Proposition 1. [1] (Proposition 4) Let $X$ be a compact space and $x ∈ X$. Then $\text{Hus}(x, X) ≤ ω_1$ if and only if $X \setminus \{x\}$ is linearly Lindelöf.

Since there are locally compact linearly Lindelöf spaces which are not Lindelöf [12] and [13], a compact space $X$ satisfying $\text{Hus}(x, X) ≤ ω_1$ may fail to be first countable at $x$. However, the following remains open:

Question 2. [1] Is a compact space $X$ satisfying $\text{Hus}(X) ≤ ω_1$ always first countable?

Since a compact space of uncountable tightness contains an uncountable convergent free sequence [11], we immediately get:

Proposition 3. A compact space $X$ such that $\text{Hus}(X) ≤ ω_1$ has countable tightness.

Arhangel’skii and Buzyakova pointed out in [1] (Theorem 6) that there is a positive answer to Question 2 under CH. This result can be improved as follows:

Proposition 4 (2^{κ_0} < ℵω). A compact space $X$ satisfying $\text{Hus}(X) ≤ ω_1$ is first countable.

2010 Mathematics Subject Classification. 54A25, 54A35, 54D55.

Key words and phrases. Compact spaces, countable tightness, c-sequential, sequential, Hušek number, cardinality.

¹ The research that led to the present paper was partially supported by a grant of the group GNSAGA of INdAM.

² The research was supported by the NSF grant No. NSF-DMS 1501506.
Proof. If \( X \) is not first countable, then there is a set \( A \subseteq X \) such that \( |A| \leq \omega_1 \) and \( \chi(p,A) \geq \omega_1 \) for some \( p \in A \) (see 6.14b in [9]). Since \( X \) is countably tight, the weight of the subspace \( A \) does not exceed \( 2^{\aleph_0} < \aleph_\omega \). Thus, \( \chi(p,A) \) is an uncountable regular cardinal \( \kappa \). Now, the compactness of \( A \) implies the existence of a sequence of length \( \kappa \) in \( A \setminus \{p\} \) converging to \( p \). As \( Hus(A) \leq Hus(X) \), we reach a contradiction. \( \square \)

A weaker question is:

**Question 5.** [1](Question 4) Let \( X \) be a compact space such that \( Hus(X) \leq \omega_1 \). Is it true that \( |X| \leq 2^{\aleph_0} \)?

Recall that a space \( X \) is tame if \( |A| \leq 2|A| \) holds for every \( A \subseteq X \) [10]. Here we call a space \( X \) countably tame if every separable subspace has cardinality at most the continuum. Of course every sequential space is tame.

**Proposition 6.** Let \( X \) be a compact space satisfying \( Hus(X) \leq \omega_1 \). If \( X \) is countably tame, then \( |X| \leq 2^{\aleph_0} \).

**Proof.** Assume by contradiction that \( |X| > 2^{\aleph_0} \). Since \( X \) has countable tightness and is countably tame, there exists a closed subspace \( Y \) satisfying \( |Y| = (2^{\aleph_0})^+ \).

Since a space is linearly Lindelöf if and only if every open cover has a subcover of countable cofinality, we see that a linearly Lindelöf space of cardinality \( (2^{\aleph_0})^+ \) has Lindelöf degree not exceeding \( 2^{\aleph_0} \). Therefore for every \( x \in Y \) we must have \( \chi(x,Y) \leq 2^{\aleph_0} \).

For each \( x \in Y \) let \( U_x \) be a base of open neighbourhoods at \( x \) satisfying \( |U_x| \leq 2^{\aleph_0} \). Since \( Y \) is countably tight and countably tame, we can construct a non-decreasing collection \( \{F_\alpha : \alpha < \omega_1 \} \) of closed subsets of \( Y \) in such a way that:

1) \( |F_\alpha| \leq 2^{\aleph_0} \) for each \( \alpha \);
2) if \( Y \setminus \bigcup V \neq \emptyset \) for a finite \( V \subseteq \bigcup \{U_x : x \in F_\alpha \} \), then \( F_{\alpha+1} \setminus \bigcup V \neq \emptyset \).

As \( Y \) has countable tightness, the set \( F = \bigcup \{F_\alpha : \alpha < \omega_1 \} \) is closed. Since \( |F| \leq 2^{\aleph_0} \), we must have \( F \neq Y \). Now, the usual closing-off argument leads to a contradiction with condition 2. \( \square \)

A space \( X \) is c-sequential [15] if for any closed set \( F \subseteq X \) and any non-isolated point \( x \in F \) there is a sequence in \( F \setminus \{x\} \) converging to \( x \).

A significant strengthening

**Proposition 7.** [1] (Theorem 13) A countably compact space \( X \) satisfying \( Hus(X) \leq \omega_1 \) is c-sequential.

In [1], page 163, the authors claimed that Martin’s Axiom implies that a compact c-sequential space is sequential. They then conclude (Corollary 14) that under Martin’s Axiom every compact space \( X \) satisfying \( Hus(X) \leq \omega_1 \) is sequential. While the latter assertion may well be true (even in ZFC), the former is false. As we will see in the next section, even CH is not enough.

2. Rančin’s problem

Rančin in [15] formulated the following:

**Question 8.** Is a compact c-sequential space sequential?
The fact that a compact space of uncountable tightness has a convergent uncountable free sequence [11] implies that a compact c-sequential space is countably tight. Hence, Rančin's question has a positive answer under PFA [2] and in some models of CH [7]. Malykhin announced in 1990 [14] the existence of a counterexample in a model satisfying \((t) + 2^\omega < 2^{\omega_1}\), but he never published this result. During the preparation of this note, he replied to a request for more information about it by saying "I left topology in 1999 and I do not remember if I have ever proved that fact". However, a much stronger counterexample (also in a model in which Martin's Axiom fails) of a compact C-closed non-sequential space is described in [5]. A space \(X\) is C-closed [8] if every countably compact subset is closed. A C-closed space is necessarily c-sequential. Here we will present a negative answer to Rančin's problem under \(\diamond\).

**Theorem 9.** \(\diamond\) implies there exists a compact c-sequential space which is not sequential.

The remainder of this section is dedicated to the proof of this theorem. We will construct a closed subset \(X\) of the uncountable product \(2^{\omega_1}\) as the inverse limit of the system \(\langle X_\alpha : \alpha \in \omega_1 \rangle\) with the usual projection maps being the bonding maps. One could think of the construction of Fedorchuk's space as a good prototype. We will ensure that \(X\) has cardinality \(\aleph_1\) and is the union of two disjoint subsets. There will be a dense countably compact subset of points of countable character. These will be identified and labelled as the points \(\{x_\alpha : \alpha \in \omega_1\}\). This set of points will be dense but proper, and since it is countably compact this ensures that \(X\) is not sequential.

The complement, call it \(Y\), in \(X\) of that dense first countable subset will be indexed as \(\{y_\alpha : \alpha \in \omega_1\}\). We will ensure that any subset of the dense first countable subset that is not compact, will have infinitely many of the \(y_\alpha\) in its closure. Also, we ensure that if \(A\) is a non-discrete subset of \(\{y_\alpha : \alpha \in \omega_1\}\) then each non-isolated point of \(A\) will be the limit of a converging sequence from \(A\). These properties ensure that \(X\) is c-sequential. Indeed, suppose that \(F \subset X\) is closed and let \(z\) be a non-isolated point of \(F\). We have to show there is a sequence from \(F\) converging to \(z\). If \(z\) has countable character, then this is obvious. This means that \(z\) is equal to \(y_\alpha\) for some \(\alpha \in \omega_1\). Also let \(A\) denote the set of \(y_\beta\) that are in \(F\). We first check that \(y_\alpha\) is a limit point of \(A\). To see this, let \(W\) be any clopen neighborhood of \(y_\alpha\), and assume for a contradiction that \(W \cap A\) is just equal to \(y_\alpha\). Since \(y_\alpha\) is a limit point of \(W \cap A\), we have that \(W \cap F \cap \{x_\beta : \beta \in \omega_1\}\) is not compact, and by assumption, has infinitely many limit points in \(A\). Finally, now that we know that \(y_\alpha\) is a limit of \(A\), we are finished by the assumption that \(A\) will have a sequence converging to \(y_\alpha\).

Let \(E\) denote the stationary set consisting of limit of limits. Let \(\{L_\xi : \xi \in \omega_1 \setminus E\}\) enumerate the infinite countable subsets of \(\omega_1\) in such a way that \(L_\xi \subset \xi\). For technical convenience we arrange that for each \(\beta \in E\) and \(\ell \in \omega\), \(L_{\beta+\ell} = \omega\). This implies CH but, in fact, we will assume that \(\diamond\) holds. In fact, suppose that there is a partition \(\{E_0, E_1, E_2\}\) of \(E\) into disjoint stationary sets, and that there is a sequence \(\{a_\alpha : \alpha \in \omega_1\}\) such that, for each \(\alpha\), \(a_\alpha\) is a subset of \(\alpha\), and for all sets
As a technical device, for each \( \beta \in \omega_1 \), let \( e_\beta \) be any bijection from \( \beta \) to \( \omega \).

We define, (as we said), \( X_\alpha \subset 2^n \), as well as, \( x_\beta^\alpha, y_\beta^\alpha \in X_\alpha \) (for \( \beta \leq \alpha \)). We also define countable sets \( \tau_\alpha \subset \alpha \) and ordinals \( \gamma_\alpha \) satisfying these inductive assumptions (the role of the \( \tau_\alpha \) are to ensure that there are converging sequences in \( Y \)). For each \( \omega \leq \beta \leq \alpha \),

1. \( X_\alpha \) is a compact subset of \( 2^n \) that projects onto \( X_\beta \),
2. \( X_\omega = 2^n \) for all \( \omega \in \omega \) and \( X_\omega = 2^n \),
3. \( \{ x^\omega_n : n \in \omega \} \) and \( \{ y^\omega_n : n \in \omega \} \) are arbitrary disjoint dense subsets of \( X_\omega \),
4. \( x_\beta^\alpha, y_\beta^\alpha \) are points in \( X_\alpha \) such that \( x_\beta^\alpha \mid \beta = x_\beta^\alpha \) and \( y_\beta^\alpha \mid \beta = y_\beta^\alpha \),
5. \( x_\beta^\alpha \) is the only point in \( X_\alpha \) that projects onto \( x_\beta^\alpha \),
6. \( \{ x_\beta^\alpha : \xi \leq \alpha \} \) and \( \{ y_\beta^\alpha : \xi \leq \alpha \} \) are disjoint and dense in \( X_\omega \),
7. if \( \beta < \alpha \), then the set \( \{ x_\beta^\alpha : \xi \in L_\beta \} \) has a limit in \( \{ x_\gamma^\alpha : \gamma \leq \alpha \} \),
8. \( \tau_\alpha \) is an infinite subset of \( \beta \) and \( \{ y_\beta^\alpha : \xi \in \tau_\alpha \} \) converges to \( y_\beta^\gamma \),
9. if \( \alpha \in E_0 \), and if the point \( \chi_\alpha \) (the characteristic function of \( a_\alpha \)) is a point of \( X_\alpha \) that is not an element of \( \{ y_\beta^\alpha : \beta < \alpha \} \), then \( x_\alpha^\alpha \) is chosen to be \( \chi_\alpha \),
10. if \( \beta \in E_1 \) and if there is a unique \( \zeta_\beta < \beta \) such that \( y_\beta^\zeta \) is in the closure of \( \{ x_\beta^\alpha : \xi \in a_\beta \} \), then, for all \( \ell \in \omega \) such that \( \beta + \ell \leq \alpha \), each of the points \( y_\beta^\alpha \) and \( y_\beta^{\alpha+\ell} \) are limits of \( \{ x_\beta^\alpha : \xi \in a_\beta \} \),

Clause (5) ensures that each \( x_\beta \) is a \( G_\delta \) and so, a point of countable character in \( X \). Clause (7) ensures that the subset \( \{ x_\beta : \beta \in \omega_1 \} \) is countably compact. As above, let \( Y \) denote the set \( \{ y_\beta : \beta \in \omega_1 \} \). Now we show that \( X \setminus Y = Y \) is the set \( \{ x_\beta : \beta \in \omega_1 \} \). Let \( x \) be any point of \( X \setminus Y \), and let \( A \) denote the set of values \( \xi \in \omega_1 \) such that \( x(\xi) = 1 \). In other words, \( x \) is the characteristic function of \( A \). Recall that \( E_0(A) \) is a stationary subset of \( E_0 \) and this is the set of \( \delta \in E_0 \) such that \( a_\delta = A \cap \delta \). Since \( x \in X \) we have that \( x \upharpoonright \delta \) is a point of \( X_\delta \) for all \( \delta \). Assume there is a \( \delta \in E_0(A) \) such that \( x \upharpoonright \delta \) is not an element of \( \{ y_\beta^\alpha : \beta < \delta \} \). By property (9), we then have that \( x_\delta = x \upharpoonright \delta \), and then by property (5), \( x = x_\delta \). So suppose there is no such \( \delta \). Then, for each \( \delta \in E_0(A) \), there is \( \beta_\delta < \delta \) such that \( x \upharpoonright \delta = y_{\beta_\delta}^\delta \). By the pressing down lemma, there is (essentially) a single such \( \beta \). But then it follows that \( x = y_{\beta_\delta} \).

Now we just have to prove those two properties of \( Y \) described in the third paragraph. First, suppose that \( A \subset \omega_1 \) satisfies that \( \{ x_\alpha : \alpha \in A \} \) is a closed but not compact subset of \( X \setminus Y \). Of course this means that there is a \( \beta \in \omega_1 \) such that
$y_\beta$ is a limit point of $\{x_\alpha : \alpha \in A\}$. We have to prove that this set has infinitely many limit points in $Y$. In fact, by intersecting with a clopen neighborhood of $y_\beta$, it is easy to see that it suffices to prove that it has more than one limit. We leave as an exercise that there is a cub $C$ such that it is easy to see that it suffices to prove that it has more than one limit. We leave the unique limit point in $\beta \delta$.

Assume that step.

Finally we consider a subset $A$ of $\omega_1$ such that $\{y_\alpha : \alpha \in A\}$ is not discrete. Fix any $\beta \in \omega_1$ such that $y_\beta$ is a limit. Again, it is a basic exercise to show that there is a cub $C$ such that for all $\delta \in C$, $y_\delta \in \delta \delta$ is a limit point of $\{y_\beta : \xi \in \delta\}$ of the set $\{x_\alpha : \alpha \in A \cap \delta\}$. For uniqueness we just need witnessing basic clopen neighborhoods with support below $\delta$ for each $\beta \neq \xi \in \delta$. Choose any $\delta \in E_1(A) \cap C$. Property (10) ensures that $\{x_\xi : \xi \in a_\delta \subset A\}$ has infinitely many limits in $Y$.

Finally we consider a subset $A$ of $\omega_1$ such that $\{y_\alpha : \alpha \in A\}$ is not discrete. Fix any $\beta \in \omega_1$ such that $y_\beta$ is a limit. Again, it is a basic exercise to show that there is a cub $C$ such that for all $\delta \in C$, $y_\delta \in \delta \delta$ is a limit point of $\{y_\beta : \xi \in \delta\}$ of the set $\{x_\alpha : \alpha \in A \cap \delta\}$. For uniqueness we just need witnessing basic clopen neighborhoods with support below $\delta$ for each $\beta \neq \xi \in \delta$. Choose any $\delta \in E_1(A) \cap C$. Property (10) ensures that $\{x_\xi : \xi \in a_\delta \subset A\}$ has infinitely many limits in $Y$.

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Now it remains to carry out the induction. We can use this next lemma in each step.

**Lemma 10.** Assume that $X$ is a compact 0-dimensional metric space. Let $z$ be any non-isolated point of $X$. Assume that $\{\sigma_n : n \in \omega\}$ are sequences that converge to $z$, and that $\{\tau_n : n \in \omega\}$ are sets that have $z$ as a limit. Further assume that for each $n, m \in \omega$, $z$ is a limit of $\tau_n \cup \{\sigma_k : k < m\}$. Then there is a partition $U, W$ of $X \setminus \{z\}$ such that for each $n \in \omega$, $\sigma_n$ is almost contained in $U$, and for each $n, m, z$ is a limit point of each of $U \cap \tau_n \cup \{\sigma_k : k < m\}$ and $W \cap \tau_n \cup \{\sigma_k : k < m\}$.

**Proof.** Let $\{B_\ell : \ell \in \omega\}$ enumerate the family of all compact open subsets of $X \setminus \{z\}$. For convenience, let $A_\ell$ denote the union of the family $\{B_\ell : k \leq \ell\}$. Another assumption that we make for convenience is that we assume that the family of sequences $\{\sigma_k : k \in \omega\}$ is increasing. We will recursively choose a sequence $\{\ell_k : k \in \omega\}$ so that the sequence $\{B_{\ell_k} : k \in \omega\}$ are pairwise disjoint and converge to $z$. That is, if $W$ is the union of any infinite subsequence of this sequence then we will have that $U = X \setminus (\{z\} \cup W)$ will be open. Choose $\ell_0$ to be minimal such that $B_{\ell_0}$ meets $\tau_0$ and is disjoint from $\sigma_0$. At stage $k$, we choose $\ell_k$ to be minimal so that

1. $B_{\ell_k}$ is disjoint from $A_{\ell_k-1}$,
2. $B_{\ell_k}$ is disjoint from $\sigma_k$
3. $B_{\ell_k}$ meets $\tau_j$ for each $j \leq k$.

To see there is such a value $\ell$, we just note that $z$ is a limit of each of the sets $\tau_j \setminus (\sigma_k \cup A_{\ell_k-1})$. For each such $j \leq k$, choose a point $z_j^k$ from each of these sets, and there is an $\ell$ such that $\{z_j^k : j \leq k\} \subset B_\ell$ while $B_\ell$ is disjoint from $\sigma_k \cup A_{\ell_k-1}$.

Finally, set $W$ equal to $\bigcup \{B_{\ell_k} : k \in \omega\}$. By construction $W$ is almost disjoint from each $\sigma_n$. Additionally, $W$ meets $\tau_n \setminus (\sigma_k \cup A_k)$ for each pair $n, k$ and so $W \cap \tau_n \setminus \sigma_k$ has $z$ in its closure. It follows similarly that $U \cap \tau_n \setminus \sigma_k$ has $z$ in its closure for each $n, k$. □
Now we show how to select \( \{x_\beta^\alpha : \beta \leq \alpha\} \), \( \{y_\beta^\alpha : \beta \leq \alpha\} \) and \( \tau, \gamma \) depending on the value of \( \omega \leq \alpha \in \omega_1 \). Let \( \delta \) denote the largest limit such that \( \delta \leq \alpha \) and let \( \ell \in \omega \) be fixed so that \( \alpha = \delta + \ell \). If \( \delta = \alpha \), then let \( X_\alpha \) denote the intersection of the family \( \{X_\beta \times 2^{\alpha_\beta} : \beta < \alpha\} \) (i.e. the inverse limit). Also, for each \( \beta < \alpha \), let \( x_\beta^\alpha, y_\beta^\alpha \) denote the unique points in \( X_\alpha \) satisfying that \( x_\beta^\alpha \mid \gamma = x_\beta^\gamma \) and \( y_\beta^\alpha \mid \gamma = y_\beta^\gamma \) for each \( \beta < \gamma < \alpha \).

We proceed in cases:

**Case 1.1:** \( \alpha = \delta \notin E \). Clearly items (9)-(11) do not apply in this case. Since the closure of \( \{y_\alpha^\alpha : n \in \omega\} \) maps onto \( X_\omega \), we can choose a point \( y_\alpha^\alpha \notin \{x_\beta^\alpha : \beta < \alpha\} \) in this closure. Also choose \( \tau, \gamma \) so that \( \{y_\alpha^\alpha : n \in \tau\} \) converges to \( y_\alpha^\alpha \), and set \( \gamma = \alpha \). Similarly we can choose \( x_\alpha^\alpha \in X_\alpha \) simply so that it is not in \( \{y_\alpha^\alpha : \beta < \alpha\} \).

Now we verify the inductive conditions (1)-(8). Items (1)-(4) and item (6) are immediate. Item (5) holds by the induction hypothesis and because we are at a limit step. Item (7) is vacuous, and \( \tau \) was chosen so that (8) holds when we set \( \gamma = \alpha \).

**Case 1.2:** If \( 0 < \ell \), then let \( \beta \) be the predecessor of \( \alpha \). Clearly we have already chosen points \( x_\beta^\gamma, y_\beta^\gamma \) in \( X_\beta \) for all \( \xi \leq \beta \). Since \( \delta \notin E \), the main task is to ensure item (7). If \( \{x_\xi^\beta : \xi \in L_\beta\} \) has a limit point \( z \) that is not in \( \{y_\xi^\beta : \xi \leq \beta\} \), then the construction is trivial. We let \( X_\alpha = X_\beta \times \{0\} \) and, for all \( \xi \leq \beta \), both \( x_\xi^\alpha \) and \( y_\xi^\alpha \) are the unique points of \( X_\alpha \) that projects onto \( x_\xi^\beta, y_\xi^\beta \) respectively. We let \( x_\alpha^\alpha \) denote the unique point of \( X_\alpha \) that projects onto \( z \). The choice of \( y_\alpha^\alpha \) is again taken to be any limit point (not among \( \{x_\xi^\beta : \xi \leq \alpha\} \)) of \( \{y_\alpha^\alpha : n \in \omega\} \) and we let \( \tau \subset \omega \) be chosen so that \( \{y_\alpha^\alpha : n \in \tau\} \) converges to \( y_\alpha^\alpha \). Set \( \gamma = \alpha \). The verification of the inductive conditions proceeds as in Case 1.1.

So now assume that \( z \) is in the set \( \{y_\xi^\beta : \xi \leq \beta\} \). It is possible that \( z \) is the unique limit of the set \( \{x_\xi^\beta : \xi \in L_\beta\} \) and so we must “double” the point \( z \) before assigning a value to \( x_\alpha^\alpha \). Let \( \{\sigma \alpha : n \in \omega\} \) enumerate the (possibly finitely) many sequences of the form \( \{y_\xi^\beta : \xi \in \tau\} \) \( (\gamma \leq \beta) \) that converge to \( z \). Notice that \( \{x_\xi^\beta : \xi \in L_\beta\} \) is disjoint from each \( \sigma \alpha \). Apply Lemma 10 to choose the open subsets \( U \) and \( W \) of \( X_\beta \setminus \{z\} \) as indicated in the conclusion of the Lemma, namely that \( W \cap \{x_\xi^\beta : \xi \in L_\beta\} \) has \( z \) as a limit point, and that \( U \) mod finite contains \( \sigma \alpha \) for each \( n \) as well as having that \( z \) is a limit of \( U \cap \{x_\xi^\beta : \xi \in L_\beta\} \). We define \( X_\alpha \) to be \( (U \cup \{z\}) \times \{0\} \cup (W \cup \{z\}) \times \{1\} \) as a subspace of \( 2^\omega \). We define \( y_\xi^\beta \) to equal \( z^{-1} \) and we let \( x_\alpha^\beta \) be \( z^{-1} \). Similarly, \( y_\xi^\beta \) is equal to \( y_\alpha^\alpha \) for any \( \xi < \alpha \) such that \( y_\xi^\beta \) is equal to \( z \). Evidently, every point of \( X_\beta \setminus \{z\} \) has a unique extension in \( X_\alpha \), hence the definition of \( x_\xi^\beta \) for all \( \xi < \alpha \) and similarly for all \( y_\xi^\beta \neq z \). By the induction assumption (6), we can choose a sequence \( \sigma \alpha \subset \alpha \) so that \( \{y_\xi^\beta : \xi \in \tau\} \) converges to \( y_\alpha^\alpha \) as required in item (8), and set \( \gamma = \alpha \).

**Case 2.1:** \( \delta = \alpha \in E_0 \). In this case we have already defined \( X_\alpha \) and all the points in \( \{x_\beta^\alpha, y_\beta^\alpha : \xi < \alpha\} \). If \( x_\alpha \in X_\alpha \) as described in item (9), then \( x_\alpha \) is equal to \( \chi_{\alpha_{\alpha_{\alpha}}} \). Otherwise, we let \( x_\alpha \) be any point of \( X_\alpha \setminus \{y_\alpha^\alpha : \xi < \alpha\} \). Next, let \( y_\alpha \) be any point of \( X_\alpha \setminus \{x_\beta^\alpha : \xi \leq \alpha\} \) and choose \( \tau \subset \alpha \) so that \( \{y_\xi^\beta : \xi \in \tau\} \) converges to \( y_\alpha^\alpha \).
Case 2.2: $\delta \in E_0$ and $\delta < \alpha$. There are few requirements for this case. Let $\alpha = \beta + 1$ and set $X_\alpha$ equal $X_\beta \times \{0\}$. For each $\xi < \alpha$ the definitions of $x^\alpha_\xi$ and $y^\alpha_\xi$ is immediate. Choose $x^\alpha_\alpha, y^\alpha_\alpha$ distinct points of $X_\alpha \setminus \{x^\alpha_\xi, y^\alpha_\xi : \xi < \alpha\}$. Finally, choose $\tau_\alpha \subset \omega$ so that $\{y^\alpha_\xi : \xi \in \tau_\alpha\}$ converges to $y^\alpha_\alpha$.

Case 3: $\delta$ is in $E_1$ and there is a unique $\zeta < \delta$ such that $y^\delta_\zeta$ is in the closure of $\{x^\delta_\xi : \xi \in a_\delta\}$. If $\alpha = \delta$, then let $y^\alpha_\alpha$ be equal to $y^\delta_\zeta$ and also let $\tau_\alpha = \tau_\zeta$. Choose any $x^\alpha_\alpha$ in $X_\alpha \setminus \{y^\alpha_\xi : \xi \leq \alpha\}$. If $\alpha = \beta + 1$, then let $z$ denote $y^\delta_\zeta$ and apply Lemma 10 to choose disjoint open $U, W$ so that $\{y^\delta_\xi : \xi \in \tau_\alpha\}$ is almost contained in $U$ for all $\gamma < \alpha$ such that $y^\gamma_\gamma = y^\alpha_\alpha$. Also, by Lemma 10, ensure that $z$ is a limit of each of $U \cap \{x^\delta_\xi : \xi \in a_\delta\}$ and $W \cap \{x^\delta_\xi : \xi \in a_\delta\}$. Define $X_\alpha$ to equal $((U \cup \{z\}) \times \{0\}) \cup ((W \cup \{z\}) \times \{1\})$ and set $y^\alpha_\alpha = z^0$ and $y^\alpha_\gamma = z^{-1}$. Choose $\tau_\alpha \subset \alpha$ as usual, as well as $x^\alpha_\alpha$ in $X_\alpha \setminus \{y^\alpha_\xi : \xi \leq \beta\}$. By our assumption that $L_\beta$ is equal to $\omega$, item (7) is immediate.

Case 4. $\delta \in E_2$. For easier reference we restate the key requirements for this case:

1. if $\alpha < \beta + e_\delta(\gamma)$, then $y^\alpha_\delta$ is still a limit point of $\{y^\alpha_\xi : \xi \in a_\delta\}$, and
2. if $\beta = \delta + e_\delta(\gamma) \leq \alpha$, then $\gamma_\beta = \gamma$ and $\tau_\beta \subset a_\delta$ (and by clause 8) $\{y^\beta_\xi : \xi \in \tau_\beta\}$ converges to $y^\alpha_\delta$.

If $\alpha = \delta$, we have already defined $X_\alpha$. Otherwise, choose $\beta$ so that $\alpha = \beta + 1$, and define $X_\alpha$ to be $X_\beta \times \{0\}$. For all $\gamma < \alpha$, define $x^\alpha_\gamma$ and $y^\alpha_\gamma$ in the obvious way. We have clearly preserved the inductive requirement that $\{y^\alpha_\xi : \xi \in \tau_\alpha\}$ converges to $y^\alpha_\alpha$ for all $\xi < \alpha$. It is also immediate that we have preserved that $y^\alpha_\alpha$ is a limit of $\{y^\alpha_\gamma : \xi \in a_\delta\}$ for any $\gamma < \alpha$, define $x^\alpha_\gamma$ and $y^\alpha_\gamma$ in the obvious way. We have clearly preserved the inductive requirement that $\{y^\alpha_\xi : \xi \in \tau_\alpha\}$ converges to $y^\alpha_\gamma$ for all $\xi < \alpha$. It is also immediate that we have preserved that $y^\alpha_\gamma$ is a limit of $\{y^\alpha_\gamma : \xi \in a_\delta\}$ for any $\gamma < \delta$ such that $y^\alpha_\gamma$ was a limit of $\{y^\gamma_\xi : \xi \in a_\delta\}$ for any $\gamma < \delta$. Choose $\gamma < \alpha$ so that $e_\delta(\gamma) = \ell$. We have, by induction assumption, that $y^\alpha_\gamma$ is a limit point of $\{y^\alpha_\xi : \xi \in a_\delta\}$, so choose $\tau_\alpha \subset a_\delta$ so that $\{y^\alpha_\xi : \xi \in \tau_\alpha\}$ converges to $y^\alpha_\gamma$ and set $\gamma_\alpha = \gamma$. Choose $y^\alpha_\alpha \in X_\alpha \setminus \{x^\alpha_\xi : \xi < \alpha\}$ arbitrarily. Similarly choose $x^\alpha_\alpha \in X_\alpha \setminus \{x^\alpha_\xi : \xi < \alpha\}$.

This completes the proof of Theorem 9.

3. ONE MORE REMARK

Recall that a space $X$ is said to be weakly Whyburn provided that for any non-closed set $A$ there is a set $B \subseteq A$ such that $|B \setminus A| = 1$. Clearly, a space is sequential if and only if it is weakly Whyburn and $c$-sequential.

A space $X$ is pseudoradial if for any non-closed set $A$ there is a well-ordered net $S \subseteq A$ converging to a point outside $A$. In [3] it was observed that any compact weakly Whyburn space is pseudoradial. Much harder it is to show that the previous implication is not reversible [4] (Theorem 2.3). The space we constructed in Theorem 9 is sequentially compact, being a compact space of cardinality $\aleph_1$. Since the continuum hypothesis implies that a compact sequentially compact space is pseudoradial [16], we obtain another example of a compact pseudoradial non weakly Whyburn space. This new example is in addition $c$-sequential and of size $\aleph_1$.

Notice that, the one-point compactification of Ostaszewski’s space provides a compact weakly Whyburn (hence pseudoradial) space of countable tightness which is not $c$-sequential.
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