(1) Let $a = 4 - 3i$ and $b = -2 + 5i$. Compute each expression indicated. If the answer is not a real number then write it in the form $x + yi$. Leave expressions like $\sqrt{3}$ and $\cos 2\theta$ unevaluated.

(a) $\Re a = 4$

(b) $\Im a = -3$

(c) $\bar{a} = 4 + 3i$

(d) $5a = 20 - 15i$

(e) $a + b = (4 - 2) + (-3 + 5)i = 2 + 2i$

(f) $\exp a = e^{4 \cos(-3)} + i e^{4 \sin(-3)}$

(g) $|a| = \sqrt{4^2 + (-3)^2} = 5$

(h) $ab = 4(-2) + 4(5i)(-3i)(-2) + (-3i)(5i)$

$= -8 + 20i + 6i - 15i^2 = -(8 + 15) + (20 + 6i) = 7 + 26i$

(i) $\arg a = \arctan(-\frac{3}{4})$ [ $\phi$ in the diagram ]

(j) $\frac{a}{b} = \frac{(4 - 3i)}{(-2 + 5i)}$

$= \frac{(4 - 3i)(-2 - 5i)}{(-2 + 5i)(-2 - 5i)}$

$= \frac{-23 - 14i}{4 + 25}$

$= \frac{-23}{29} - \frac{14}{29}i$
(2) State the definitions of \( \sin z \) and \( \cos z \) and use them to show that for any \( z \in \mathbb{C}, \sin^2 z + \cos^2 z = 1 \). (Do not use \( z = x + yi \). Quote properties of any function that you know except those of \( \sin z \) and \( \cos z \).)

\[
\sin z = \frac{1}{2i} (e^{iz} - e^{-iz}) \quad \cos z = \frac{1}{2} (e^{iz} + e^{-iz})
\]

\[
\sin^2 z + \cos^2 z = -\frac{1}{4} (e^{2iz} - 2e^0 + e^{-2iz})
\]
\[
+ \frac{1}{4} (e^{2iz} + 2e^0 + e^{-2iz})
\]
\[
= \frac{2}{4} + \frac{2}{4} = 1
\]

(3) Find all solutions of each equation. Write the answer in the form \( x + yi \) or \( re^{i\theta} \), whichever is easier.

(a) \( \frac{2z}{z + 2 - i} = -3 + 4i \)

\[
2z = (-3 + 4i)z + (-3 + 4i)(2 - i)
\]
\[
(5 - 4i)z = -2 + 11i
\]
\[
z = \frac{-2 + 11i}{5 - 4i} = \frac{(-2 + 11i)(5 + 4i)}{(5 - 4i)(5 + 4i)} = \frac{-54 + 47i}{25 + 16}
\]
\[
= -\frac{54}{41} + \frac{47}{41}i
\]

(b) \( z^5 - 2 + 2i = 0 \)

\[
z^5 = 2 - 2i = \sqrt[5]{8} e^{-\frac{\pi}{4}i} \quad \text{[see diagram]}
\]
Since \( \sqrt[5]{8} = 10^{\frac{1}{5}} \)

\[
z_0 = 10^{\frac{1}{5}} \exp \left( -\frac{\pi}{20}i \right)
\]
\[
z_1 = 10^{\frac{1}{5}} \exp \left( \left( -\frac{\pi}{20} + \frac{2\pi}{5} \right)i \right)
\]
\[
z_2 = 10^{\frac{1}{5}} \exp \left( \left( -\frac{\pi}{20} + \frac{4\pi}{5} \right)i \right)
\]
\[
z_3 = 10^{\frac{1}{5}} \exp \left( \left( -\frac{\pi}{20} + \frac{6\pi}{5} \right)i \right)
\]
\[
z_4 = 10^{\frac{1}{5}} \exp \left( \left( -\frac{\pi}{20} + \frac{8\pi}{5} \right)i \right)
\]

Remark: you could use \( \frac{7\pi}{4} \) for the argument of \( 2 - 2i \) and compute 5 roots using it. If you simplify \( -\frac{\pi}{20} + \frac{2\pi}{5} = \frac{7\pi}{20} \) and so on, you see that you have the same 5 complex numbers.
(4) State the definition of the derivative of \( f(z) \) and use it to find the derivative of \( f(z) = 1/z \). (You need not justify your answer with an \( \varepsilon-\delta \) argument.)

\[
f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}
\]

In this case:

\[
f'(z) = \lim_{h \to 0} \frac{\left[ (\frac{1}{z+h}) - \frac{1}{z} \right]}{h} = \lim_{h \to 0} \frac{\left[ -\frac{h}{(z+h)z} \right]}{h}
\]

\[
= \lim_{h \to 0} \frac{-1}{(z+h)z} = -\frac{1}{z^2}.
\]

(5) Find each derivative (without using limits). No need to simplify.

(a) \( \frac{d}{dz}((-5+7i)z^2) = 2(-5+7i)z \)

(b) \( \frac{d}{dz}\left(\left(\frac{(2-9i)z+i}{iz-12}\right)^6\right) = 6\left(\frac{(2-9i)z+i}{iz-12}\right)^5 \frac{(iz-12)(2-9i)-(iz-12)^2}{(iz-12)^2} \).

(c) \( \frac{d}{dz}(\sin^2 z^2) = 2 \sin z^2 \cdot \cos z^2 \cdot 2z \)

(6) Find the set of points in \( \mathbb{C} \) at which \( f(x+yi) = (x^2 + y) + i(y^2 - x) \) is differentiable.

\[
U(x,y) = x^2 + y \\
V(x,y) = -x + y^2 \\
U_x = 2x \\
V_x = -1 \\
U_y = 1 \\
V_y = 2y
\]

The partial derivatives exist and are continuous everywhere, hence \( f'(z) \) exists exactly where the Cauchy-Riemann equations hold.

\[
U_y = -V_x \text{ holds everywhere} \\
U_x = 2x = 2y = V_y \text{ holds if and only if } x = y.
\]

Thus \( f \) is differentiable along \( x=y \) and nowhere else.

\[
(f'(x+yi) = U_x(x,y) + V_x(x,y)i = 2x - i).
\]
Compute \( \int_C z^2 + \overline{z}^2 \, dz \) where \( C \) is the straight line segment from 1 to \( i \). Hint: first write \( z \) as \( x + yi \) and simplify \( z^2 + \overline{z}^2 \); factor constants out of the integrand.

\[
\overline{z} + z = (x+y \, i)^2 + (x-y \, i)^2 = x^2 + 2xy \, i - y^2 + x^2 - 2xy \, i - y^2 = 2x^2 - 2\, y^2 = 2(x^2 - y^2)
\]

Parametrize \( C \) by \( z(t) = x(t) + y(t) \, i = (1-t) + t \, i \), \( 0 \leq t \leq 1 \) so that \( z'(t) = x'(t) + y'(t) \, i = -1 + i \).

Then
\[
\int_C z^2 + \overline{z}^2 \, dz = \int_0^1 \sqrt{2 \left( (1-t)^2 - t^2 \right)} \left( -1 + i \right) \, dt
\]
\[
= 2(-1+i) \int_0^1 \sqrt{1-2t+t^2} \, dt
\]
\[
= 2(-1+i) \int_0^1 \sqrt{t} \, dt
\]
\[
= 2(-1+i) \left[ \frac{t^{3/2}}{3/2} \right]_0^1
\]
\[
= 0
\]

(8) Suppose \( f(z) = f(x+yi) \) is entire and has the form \( f(x+yi) = u(x) + v(y)i \). Show that there exist constants \( c_1 \) and \( c_2 \) such that \( f(z) = c_1 z + c_2 \).

Since \( f' \) exists everywhere the Cauchy-Riemann equations hold for all \( x \) and \( y \):

\[
u = u'(x) = v'(y) = u_y \quad \text{and} \quad u_y = 0 = -v_x
\]

The second equation is automatic. The first can hold only if \( u'(x) \) and \( v'(y) \) are equal to the same real constant, call it \( a \). We find \( u(x) \) and \( v(x) \) by integration.

\[
u(x) = \int u'(x) \, dx = \int a \, dx = ax + b \, , \text{some} \, b \in \mathbb{R}
\]

\[
u(y) = \int v'(y) \, dy = \int a \, dy = ay + c \, , \text{some} \, c \in \mathbb{R}
\]

Thus

\[
f(z) = u(x) + v(y)i = (ax + b) + (ay + c)i
\]

\[
= a(x + yi) + (b + ci)
\]

\[
= c_1z + c_2 \quad \text{where} \quad c_1 = a \quad \text{and} \quad c_2 = b + ci.
\]