Name: Solutions

Do not evaluate expressions like \( \sin e^{-2} \) or simplify expressions like \( 1/(2 + 3i) \).

(1) State the definition of each function (use \( z = x + yi \) or \( z = re^{i\theta} \) as appropriate).

(a) \( \log z = \log re^{i\theta} = \ln r + i\theta \)

(b) For \( z \neq 0 \) and \( a \notin \mathbb{Z} \), \( z^a = \exp(a \log z) \)

(2) Compute all possible values of each expression.

(a) \( \log(\sqrt{2} - i\sqrt{2}) \)

\[ \sqrt{2} - i\sqrt{2} = 2e^{i(-\pi/4 + 2k\pi)}i, \quad k \in \mathbb{Z} \]

\[ \text{so} \]

\[ \log \sqrt{2} - i\sqrt{2} \]

\[ = \ln 2 + i(-\pi/4 + 2k\pi)i, \quad k \in \mathbb{Z} \]

(b) \( 5^i \)

\[ 5 = 5e^{i(0 + 2k\pi)}, \quad k \in \mathbb{Z} \]

\[ \text{so} \]

\[ 5^i = \exp(i \log 5) \]

\[ = \exp(i [\ln 5 + 2k\pi i]) \]

\[ = \exp(-2k\pi + \ln 5) \]

\[ = e^{-2k\pi} (\cos \ln 5 + i\sin \ln 5), \quad k \in \mathbb{Z} \]
(3) Compute each integral, under the assumption that all simple closed curves are positively oriented. Fully explain your work, quoting the theorems that you use. If you reduce a problem to the computation of a real integral, you may stop at that point for full credit, or continue for possible extra credit.

(a) \( \int_C z \sqrt{1 + z^2} \, dz \), \( C \) the straight line segment running from 1 to i

\[
\frac{d}{dz} \left[ (1+z^2)^{\frac{3}{2}} \right] = \frac{3}{2} (1+z^2)^{\frac{1}{2}} \cdot 2z = 3z \sqrt{1+z^2}
\]

hence

\[
z \sqrt{1+z^2} = \frac{d}{dz} \left[ \frac{1}{3} (1+z^2)^{3/2} \right] \text{ all along } z
\]

no matter what branch of logarithm is used to define the powers.

Thus setting \( F(z) = \frac{1}{3} (1+z^2)^{3/2} \),

\[
\int_C z \sqrt{1+z^2} \, dz = F(i) - F(1) = 0 - \frac{1}{3} \cdot 2^{3/2} = -\frac{1}{3} \sqrt{8},
\]

by the Fundamental Theorem.

(b) \( \int_C \sin^2 z^2 \, dz \), \( C \) the circle of radius 4, centered at 1 + i

Since \( \sin^2 z^2 \) is entire and \( C \) is a simple closed curve, by the Closed Curve Thm (or Cauchy-Goursat Thm)

\[
\int_C \sin^2 z^2 \, dz = 0.
\]
(c) \( \int_{C} \frac{\cos z}{(z-4)^2(z-1)} \, dz \), \( C \) the circle of radius 2, centered at 2 + i

\[ |4 - (2 + i)| = |2 - i| = \sqrt{4 + 1} = \sqrt{5} > 2 \]
so 4 lies outside the circle

\[ |1 - (2 + i)| = |-1 - i| = \sqrt{1 + 1} = \sqrt{2} < 2 \]
so 1 lies inside the circle

Apply the Cauchy Integral Formula:

\[ \int_{C} \frac{\cos z}{(z-4)^2(z-1)} \, dz = \int \left[ \frac{\cos z}{(z-4)^2} \right] \, dz = 2\pi i \left[ \frac{\cos z}{(z-4)^2} \right] \bigg|_{z=1} \]

\[ = \frac{2\pi i \cdot \cos 1}{9} \]

**Remark:** \( z \) is not analytic
so no theorem applies; we must parametrize.

(d) \( \int_{C} z |z| \, dz \), \( C \) the arc running from 2 to 2i of the circle of radius 2

centered at 0

Parametrize \( C \) by

\[ z(t) = 2 \cos t + i \sin t, \ 0 < t < \frac{\pi}{2} \]
\[ z'(t) = -2 \sin t + 2i \cos t \]

\[ \int_{C} |z| |z| \, dz = \int_{0}^{\pi/2} (2 \cos t + 2i \sin t) \cdot 2 \cdot (-2 \sin t + 2i \cos t) \, dt \]

\[ = 8 \int_{0}^{\pi/2} - \cos t \sin t - i \sin^2 t + i \cos^2 t - \cos t \sin t \, dt \]

\[ = -16 \int_{0}^{\pi/2} \sin t \cos t \, dt + 8i \int_{0}^{\pi/2} \cos^2 t - \sin^2 t \, dt \quad \text{[full credit]} \]

\[ = [8 \cos^2 t]_{0}^{\pi/2} + 8i \int_{0}^{\pi/2} \cos 2t \, dt \]

\[ = -8 + 4i \left[ \sin 2t \right]_{0}^{\pi/2} \]

\[ = -8 \]
(4) Find the unique power series expansion of each function in powers of \( z - z_0 \), and its radius of convergence.

(a) \( f(z) = z/(z+2) \), \( z_0 = 0 \).

Use the geometric series:
\[
\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \quad \text{for} \quad |r| < 1.
\]

\[
f(z) = \frac{z}{z+2} = \frac{1}{2} \cdot \frac{1}{\frac{1}{2} \cdot \frac{1}{z}} = \frac{1}{2} \cdot \frac{1}{1-\left(-\frac{z}{2}\right)}
\]

\[
= \frac{1}{2} \cdot z \sum_{k=0}^{\infty} \left(-\frac{z}{2}\right)^k \quad \text{valid for} \quad |\frac{z}{2}| < 1
\]

\[
= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{2^k+1} \cdot z^k \quad \text{valid for} \quad |z| < 2
\]

Remark: this can also be done using the identity
\[
\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k,
\]
valid for \(|r| < 1\).

See the extra page adjoined to the end of these solutions.

(b) \( f(z) = 1/z \), \( z_0 = 1 \).

Use the Taylor Series:
\[
\sum_{k=0}^{\infty} \frac{1}{k!} \cdot f^{(k)}(z_0) \cdot (z-z_0)^k
\]

\[
f(z) = z^{-1} \quad f'(1) = 1 = 0!
\]

\[
f'(z) = -z^{-2} \quad f''(1) = -1 = -1!
\]

\[
f'' (z) = 2z^{-3} \quad f'''(1) = 2 = 2!
\]

\[
f'''(z) = -6z^{-4} \quad f^{(4)}(1) = -6 = 3!
\]

\[
f^{(4)}(z) = 24z^{-5} \quad f^{(5)}(1) = 24 = 4!
\]

\[
\vdots
\]

\[
f^{(k)}(z) = (-1)^k \cdot k! \cdot z^{-k-1} \quad f^{(k)}(1) = (-1)^k \cdot k!
\]

Thus
\[
f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot k! \cdot (-1)^k \cdot (z-1)^k = \sum_{k=0}^{\infty} (-1)^k \cdot (z-1)^k
\]

Ratio Test:
\[
\lim_{k \to \infty} \left| \frac{(-1)^{k+1}(z-1)^{k+1}}{(-1)^k(z-1)^k} \right| = \lim_{k \to \infty} |z-1| < 1 \quad \text{if} \quad |z-1| < 1
\]

The radius of convergence is 1.
(5) For a fixed $y_0 \in \mathbb{R}$ let $S = \{z : y_0 < \text{Im}z < y_0 + 2\pi\}$. Show that the branch of logarithm taking values in $S$ is the inverse of $e^z$ restricted to $S$.

$$W = e^z$$

We must show: (a) $\log e^z = z \forall z \in S$ and (b) $e^{\log w} = w \forall w \in \text{(slit plane)}$.

(a) If $z \in S$ then $z = x + iy$, $y_0 < y < y_0 + 2\pi$, and

$$\log (e^z) = \log (e^{x+iy}) = \log (e^x e^{iy})$$

But $e^x$ is playing the role of $r$ and, since $y_0 < y < y_0 + 2\pi$, $y$ is playing the role of the unique choice of $\theta$ for this branch of $\log$, so

$$\log (e^x e^{iy}) = \ln e^x + i y = x + i y = z$$

as required.

(b) For $w \in \mathbb{C}$ (not on the ray, or else include one edge of $S$, say $y = y_0$), $w = re^{i\theta}$ for unique $r > 0$ and unique $\theta$ such that $y_0 < \theta < y_0 + 2\pi$, so that

$$e^{\log w} = e^{\log re^{i\theta}} = e^{\ln r + i\theta}$$

Now $\ln r$ is the real part and $\theta$ is the imaginary part of the input into the exponential function, so

$$e^{(\ln r + i\theta)} = e^{\ln r} e^{i\theta} = re^{i\theta} = w$$

as required.
Alternate approach to 4(b): power series for $f(z) = \frac{1}{z}$ in powers of $z - 1$:

Use the identity $\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k$, valid for $|r| < 1$.

$$\frac{1}{z} = \frac{1}{z-1+i} = \frac{1}{1+[-(z-1)i]} = \sum_{k=0}^{\infty} [-(z-1)i]_k = \sum_{k=0}^{\infty} (-1)^k (z-1)_k$$

valid for $|z-1| < 1$, so the radius of convergence is 1.