The notation p/n means the problem with number n on page p of Perko.

1. 6/3 Due: W, Jan 18
2. 6/5 and describe the relationship of the phase portraits Due: W, Jan 18
3. 6/6 Due: W, Jan 18
4. 9/1a Due: W, Jan 18
5. 9/1b Due: W, Jan 18
6. 9/1c Due: W, Jan 18
7. 9/3a [Use the hint.] Due: W, Jan 18
8. 10/5 Prove that the conditions that you identify are necessary and sufficient. Due: W, Jan 18
9. 10/6 This is an $\epsilon$-$\delta$ proof, so your solution should begin “Fix $t \in \mathbb{R}$ and let $\epsilon > 0$ be given.” You must find an expression for $\delta$ in terms of $\epsilon$ so that if $|x_0 - y_0| < \delta$ then $|\varphi(t, x_0) - \varphi(t, y_0)| < \epsilon$. Note that you have a formula for $\varphi(t, z)$, but will have to make a number of estimates in order to get the one that you want. Due: W, Jan 18
10. Suppose $\dot{x} = \lambda x$ and $\dot{y} = \mu y$ for $\lambda, \mu > 0$. Obtain a formula of the form $x = h(y)$, $y = h(x)$, or $h(x, y) = 0$ for the solution curve through $(x_0, y_0) \neq (0, 0)$, for all such pairs $(x_0, y_0)$. Draw all qualitatively distinct phase portraits (there are three of them). Due: W, Jan 18
11. 20/6 Due: W, Jan 25
12. 20/8 Due: W, Jan 25
13. For $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ show that $exp(tJ) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$ by writing
   \[
   \begin{pmatrix} \lambda t & t \\ 0 & \lambda t \end{pmatrix} = \begin{pmatrix} \lambda t & 0 \\ 0 & \lambda t \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} := S + N
   \]
   and computing $exp(N)$ directly from the definition. Hint. You must apply a theorem. Due: W, Jan 25
14. For $x(t) = x_0e^{\lambda t} + y_0te^{\lambda t}$ and $y(t) = y_0e^{\lambda t}$, the solution to the linear system of the previous problem, prove that if $\lambda y_0 \neq 0$ then $x = \frac{x_0}{y_0}y + \frac{1}{\lambda}y \log \frac{y}{y_0}$. Due: W, Jan 25
15. Show that in polar coordinates the system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ (for any functions $P$ and $Q$, not necessarily linear) becomes
   \[
   r^2 \dot{\theta} = [xQ(x, y) - yP(x, y)]|_{x=r \cos \theta, y=r \sin \theta},
   \]
   \[
   r \dot{r} = [xP(x, y) + yQ(x, y)]|_{x=r \cos \theta, y=r \sin \theta}.
   \]
   Hint. Exercise 28/9 of Perko. Due: W, Jan 25
16. For the system $\dot{x} = Jx$, $J = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $b \neq 0$, change to polar coordinates to obtain an uncoupled system $\dot{\theta} = f(\theta)$, $\dot{r} = g(r)$. Solve the system explicitly (find $\theta(t)$ and $r(t)$) and use the solution to rigorously derive the phase portraits possible (with attention to the signs of $a$ and $b \neq 0$). Due: W, Jan 25

17. 26/1 Note that you are not to solve the system. Due W, Feb 1

18. For each saddle in 26/1 find the equations of the separatrices. (They are the eigenspaces.) Due W, Feb 1

19. 26/5 Due W, Feb 1

20. Show that if $\eta(t)$ solves $\dot{x} = f(x)$, $x(t_0) = x_0$, then there exists a constant $c$ such that $\mu(t) := \eta(t + c)$ solves $\dot{x} = f(x)$, $x(0) = x_0$. Find $c$ explicitly. Due W, Feb 1

21. Let $|\cdot|$ denote the euclidean norm on $\mathbb{R}^n$.
(a) Find $A$ and $B$ so that $A|x|_{\max} \leq |x| \leq B|x|_{\max}$.
(b) Find $A$ and $B$ so that $A|x|_{\sum} \leq |x| \leq B|x|_{\sum}$.
In each case attempt to find values of $A$ and $B$ so that equality holds for at least one value of $x$ (not necessarily the same $x$ in each of the inequalities in each pair). Due W, Feb 1

22. Suppose $E \subset \mathbb{R}$ is open, $f : E \to \mathbb{R}^n$ is continuous, and $[a, b] \subset E$. Prove that the inequality $|\int_a^b f(t) \, dt| \leq \int_a^b |f(t)| \, dt$ is valid for the norms $|x| = |x_1| + \cdots + |x_n|$ and $|x| = \max\{|x_1|, \ldots, |x_n|\}$.
Remark: for the euclidean norm $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, there is an analogous inequality with a scaling factor on the right hand side (by equivalence of norms). As a bonus problem you could see if you can either prove or find a counterexample to the claim that the scaling factor can be chosen to be 1. Due W, Feb 1

23. Suppose $f : I \times E \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and consider the initial value problem
$$\dot{y} = f(t, y), \quad y(t_0) = y_0$$
and the integral equation
$$\eta(t) = y_0 + \int_{t_0}^t f(s, \eta(s)) \, ds.$$ 
(a) Show that if $\eta : J \subset I \to E$ is continuous and satisfies (2) then $\eta$ is differentiable and satisfies (1).
(b) Show that if $\eta : J \subset I \to E$ is differentiable and satisfies (1) then the integral in (2) exists and $\eta$ satisfies (2).
Due W, Feb 1

24. a. Use the successive approximations technique to solve the initial value problem $\dot{x} = \sin x$, $x(0) = 0$. 
25. Let $\mu(t)$ be any solution to the initial value problem $\dot{y} = f(t, y)$, $y(t_0) = y_0$, on some open interval $J_1 \subset \mathbb{R}$ about $t_0$. For $M$, $K$, and the functions $\eta_j$ as in the proof of the existence/uniqueness theorem, prove by induction that (shrinking $J_1$ if necessary)

$$|\mu(t) - \eta_j(t)| \leq M \frac{K^j|t - t_0|^j}{j!}$$

for all $j \in \mathbb{N} \cup \{0\}$. Use this estimate to show that $\eta_j \to \mu$ (at least pointwise) on $J_1$, so that $\mu = \eta$ on $J_1$. Due W, Feb 8

26. Prove the existence/uniqueness theorem using the Contraction Mapping Principle. (See 78/5 of Perko.) Due W, Feb 8

27. In the proof of Lemma 1 on page 85, we assumed in a proof by contradiction that an endpoint $t_0$ of $I^*$ lay in the interval $I$. Prove that if this is so then there exists a neighborhood $N$ of $t_0$ in $I$ on which both $u_1$ and $u_2$ are defined and continuous, that both $\lim_{t \to t_0} u_1(t)$ and $\lim_{t \to t_0} u_1(t)$ exist and have a common value $u_0$, and that $u_0 \in E$. Due W, Feb 15

28. 94/2 (Maximum interval of existence.) Due W, Feb 15

29. 95/4 (change to cylindrical coordinates and study.) Due W, Feb 15

30. Suppose $f : E \subset \mathbb{R}^n \to \mathbb{R}^n$ is at least $C^1$ and that $\varphi : (\alpha, \beta) \to \mathbb{R}^n$ is a solution of the initial value problem $\dot{x} = f(x)$, $x(0) = x_0 \in E$. Let $g(x) = k f(x)$, $k \in \mathbb{R} \setminus \{0\}$. Express the solution $\eta(t)$ of the initial value problem $\dot{x} = g(x)$, $x(0) = x_0$, in terms of $\varphi(t)$, being careful about its domain. Discuss the relationship of the phase portraits of the two differential equations on $E$. (Compare with Problem 2 on this list.) Due W, Feb 15

31. Same problem as the previous problem but for $g(x) = k(x) f(x)$, where $k : \mathbb{R}^n \to \mathbb{R} \setminus \{0\}$ is at least $C^1$. Due W, Feb 15

32. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$, $f$ is at least $C^1$, and there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}^n$, then for any $x_0 \in \mathbb{R}^n$ the unique solution to the initial value problem $\dot{x} = f(x)$, $x(0) = x_0$, exists for all $t \in \mathbb{R}$. Due W, Feb 15

33. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ and is at least $C^1$, and if we are studying geometric properties of the phase portrait of the system $\dot{x} = f(x)$ of ordinary differential equations on $\mathbb{R}^n$, then we may assume without loss of generality that solutions exist on all of $\mathbb{R}$. Hint. You may use the results of the previous two problems. Due W, Feb 15

34. Duffing’s equation is $\ddot{x} - x + x^3 = 0$

a. Write an equivalent system of first order equations

$$\begin{align*}
\dot{x} & = P(x, y), \\
\dot{y} & = Q(x, y).
\end{align*}$$

b. Show that $H : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto \frac{1}{4} x^4 - \frac{1}{2} x^2 + \frac{1}{2} y^2$ is constant on trajectories, so that every level curve of $H$ is an invariant set for (*).
c. Using a symbolic manipulator like Maple or Mathematica, or by hand, find
\( H^{-1}(h) \) for any value of \( h \) for which it contains an equilibrium of (*) and for
nearby \( h \). Using (*), decompose these level curves into trajectories and add
arrows. (Thus in this part you will submit two figures.) Due W, Feb 22

35. 117/2 Make the change of notation \( x_1 \rightarrow x \) and \( x_2 \rightarrow y \). Also avoid subscripted
variables by using the notation
\[
\psi^{(j)}(t, a) = \left( u^{(j)}(t, a), v^{(j)}(t, a) \right)
\]
and \( a = (a, b) \).
Note that this is the system of the fully worked out Example 2 on page 99
so you actually have the answer in hand before you do these more involved
computations. For an example of the computations that you must do in a
specific case see Example 2 in this section of the text (page 111). (See also the
next problem for yet a third approach to finding \( W_{\text{loc}}^s(0) \) for this system.) Due
W, Feb 29

36. The approaches to finding the local stable and unstable manifolds used in Ex-
ample 2 on page 99 and in the previous problem are not the ones typically used
in practice. (It is exceedingly rare to be able to solve a system of interest, for
example.) Instead one usually proceeds as outlined in this problem, in which
you find the first few terms in the function \( y = \psi(x) \) that gives the local stable
manifold at the origin of the system
\[
(*) \quad \dot{x} = -x - y^2, \quad \dot{y} = y + x^2.
\]
a. Write \( \psi(x) = a_0 + a_1 x + ax^2 + bx^3 + cx^4 + dx^5 + \cdots \). Explain why you know
from the Stable Manifold Theorem (without doing any computation at all)
that \( a_0 = a_1 = 0 \).
b. Explain why \( \psi \) and \( \psi' \) must satisfy
\[
(**) \quad -(x + \psi^2)\psi' = x^2 + \psi.
\]
Hint. The Chain Rule enters in.
c. Expand both sides of (***) through order seven and equate terms to obtain \( \psi \)
through order seven.
Due W, Feb 29

37. Use the approach described in the previous problem to find the local stable
manifold at the origin for the system in Problem 35. Due W, Feb 29

38. In the proof of the Flowbox Theorem, \( f(0) = (k, 0, \ldots, 0)^T \) and \( g \) was defined
by
\[
g: [-\alpha, \alpha] \times N \subset \mathbb{R} \times \mathbb{R}^{n-1} \equiv \mathbb{R}^n \rightarrow \mathbb{R}^n
\]
\[
: (t, (x_2, \ldots, x_n)) \mapsto \eta(t, (0, x_2, \ldots, x_n)).
\]
By the very definition of \( g \) it is as smooth as \( f \).
a. Compute \( dg(0, 0) \). The answer is an \( n \times n \) matrix of specific constants. Hint.
The first column is \( \dot{\eta} \). For the remaining columns you may set \( t = 0 \) before
computing the first partial derivatives.
b. Apply the Inverse Function Theorem to conclude that there exists a neigh-
borhood \( V \) of \((0, 0)\) in \([-\alpha, \alpha] \times N \) and a \( C^\infty \) mapping \( h : U := g(V) \rightarrow V \)
such that (i) \( g \circ h = id_U \) and (ii) \( h \circ g = id_V \). Due W, Feb 29

39. Let \( L \) be a hyperbolic linear mapping on \( \mathbb{R}^n \) and let \( \varphi(t, x) \) denote the global
flow \( \varphi(t, x) = \exp(tL)x \) of \( \dot{x} = Lx \). Let \( C^\infty_0(\mathbb{R}^n) \) denote the set of mappings from
$\mathbb{R}^n$ to itself that are uniformly continuous and uniformly bounded, and for $\mu > 0$ let $\mathcal{L}_\mu$ denote the set of mappings of the form $\Lambda = L + \lambda$ where $\lambda \in C^0_0(\mathbb{R}^n)$ is uniformly bounded by $\mu$ and is Lipschitz with Lipschitz constant at most $\mu$. For $\mu > 0$ let $L_{\mu}$ denote the set of mappings of the form $\Lambda = L + \lambda$ where $\lambda \in C^0_0(\mathbb{R}^n)$ is uniformly bounded by $\mu$ and is Lipschitz with Lipschitz constant at most $\mu$.

For $\Lambda \in L_{\mu}$ let $\eta(t, x)$ denote the local flow generated by $\dot{x} = \Lambda(x)$. Prove that if $\mu > 0$ is sufficiently small, then for each $\Lambda \in L_{\mu}$, $\eta(t, x)$ is actually defined on $[0, \infty)$. Address the question of whether the uniform bound on $\lambda$ must be small, or if the control of the size of its Lipschitz constant suffices.

Hint. Let $T > 0$ be given. For any $t \in [0, T]$ for which $\eta(t, x)$ exists bound $|\eta(t, x) - x|$ above by some quantity involving $T$ (hence independent of $t$) using the triangle inequality vis-à-vis $\phi(t, x)$ and an integral equation that $\eta(t, x)$ satisfies on its maximal interval of existence $I(x)$. Conclude that $\eta(t, x)$ exists on $[0, T]$.

Due W, Mar 21

40. In the context of the previous problem, but with $\mu$ replaced by $\nu$ (so that the notation in this exercise matches the part of the lecture notes in which the exercise is needed) show that

$$|\eta(t, x) - \eta(t, y)| \leq e^{((||L||+\nu)t)|x - y|}.$$

Hint. Use an integral equation that $\eta$ satisfies and Gronwall’s Inequality.

Due W, Mar 21

41. Use the Hartman-Grobman Theorem to classify the singularity specified, or state why the theorem does not apply. In the case of saddle points, use the Stable Manifold Theorem to identify the tangent lines at the singularity to the stable and unstable separatrices. Due W, Mar 28

a. $\dot{x} = x(x - y), \dot{y} = y(2x - y)$ at $(0, 0)$.
b. $\dot{x} = x - x^3, \dot{y} = -2y + y^2$ at $(0, 0)$.
c. $\dot{x} = x - y, \dot{y} = 1 - e^x$ at $(0, 0)$.
d. $\dot{x} = x + e^{-y}, \dot{y} = -y$ at $(-1, 0)$.
e. $\dot{x} = y + x - x^3, \dot{y} = -y$ at each singularity.

42. Classify the singularity at the origin of the system $\dot{x} = 2x^2 - y^2, \dot{y} = 3xy$ as either stable, asymptotically stable, or unstable. (The $x$-axis is invariant and each of the half-planes it determines is a global elliptic sector.) Due W, Mar 28

43. Any two parts of 134/2. Make the change of notation $x_1 \to x$ and $x_2 \to y$. Hint. You know more than one technique for trying to work a problem like this. Due W, Mar 28

44. Any two parts of 135/5. Make the change of notation $x_1 \to x$ and $x_2 \to y$. Hint. This is a trick problem. See the hint on the previous problem. Due W, Mar 28

45. 136/7 and 136/8. Introduce new variables in such a way that there are no subscripted variables in the system on $\mathbb{R}^2$. The theorem in the book (Theorem 3 on page 131) is enough to conclude stability, but not quite enough to prove asymptotic stability. Explain why. (In fact exactly the same proof gives a result that works for us, so that here the critical point is asymptotically stable.) Due W, Apr 4.

Hint. Recall Example 4 on page 134, and ignore the author’s hint. (Not all printings of the Third Edition of the book contain the hint.)

46. 208/1 Due W, Apr 4.
47. 209/2 Due W, Apr 4.

48. 210/4 The definition of a Hamiltonian system is Definition 1 of §2.14. On \( \mathbb{R}^2 \) (our situation in this problem) it is just a system of the form (1) in that definition where \( x \) and \( y \) are replaced with the one-dimensional variables \( x \) and \( y \), namely:
\[
\dot{x} = -H_y, \quad \dot{y} = H_x.
\]
Due W, Apr 11.

49. Suppose \( \gamma \) is a periodic orbit of a smooth system \( \dot{x} = f(x) \) on an open subset of \( \mathbb{R}^2 \), that \( \Sigma \) is a section of the flow at some point of \( \gamma \), that \( s \) is a coordinate on \( \Sigma \) chosen so that \( s = 0 \) corresponds to a point of \( \gamma \), that \( P(s) \) is the Poincaré first return map on a neighborhood of 0, and that \( d(s) \) is the difference map \( d(s) = P(s) - s \). Describe the orbit structure in a neighborhood of \( \gamma \) in terms of the first non-zero derivative \( d^{(k)}(0) \) of \( d \) at 0. Argue carefully in terms of analysis, but intuitively in terms of geometry.
Hint. What is important is the parity of \( k \) and the sign of \( d^{(k)}(0) \). Due W, Apr 11.

50. Prove as a corollary to the Poincaré-Bendixson Theorem the Poincaré Annular Region Theorem:
Suppose \( \mathcal{A} \) is the diffeomorphic image of an annulus. Suppose \( f \) is a smooth vector field defined on a neighborhood of \( \mathcal{A} \) with the following properties:
(a) \( f \) points into \( \mathcal{A} \) (respectively, out of \( \mathcal{A} \)) at every point of \( \partial \mathcal{A} \);
(b) \( f \) has no critical point in \( \mathcal{A} \cup \partial \mathcal{A} \).
Then \( f \) has a cycle that lies wholly within \( \mathcal{A} \).
(We will see later that the cycle will have to enclose the hole in \( \mathcal{A} \) in its interior.)
Due W, Apr 11.

51. Suppose a smooth vector field \( f \) has two unstable cycles \( \gamma_1 \) and \( \gamma_2 \), one in the interior of the other, in its domain of definition. Show that if \( f \) has no critical point in the annular region bounded by \( \gamma_1 \) and \( \gamma_2 \) then it contains at least one stable cycle in that region. (Of course the analogous statement regarding existence of an unstable cycle is true if \( \gamma_1 \) and \( \gamma_2 \) are stable cycles.)
Hint. Use the following fact: if \( \gamma \) is an unstable cycle then there exists an annular neighborhood \( \mathcal{N} \) of \( \gamma \) such that \( f \) points outward at every point of \( \partial \mathcal{N} \).
Due W, Apr 11.

52. If in the situation of Problem 50 \( \text{div } f \) is not identically zero and is of one sign on \( \mathcal{A} \) then the cycle in \( \mathcal{A} \) is unique.
Hint. Use Theorem 2 and its corollary in Section 3.4 of Perko. (Note that \( \nabla \cdot f = \text{div } f \).) You may use the following fact, to be proved later in the course: if \( \gamma \) is a periodic orbit of \( \dot{x} = f(x) \) on an open subset \( E \) of \( \mathbb{R}^2 \) and if \( \text{Int } \gamma \subset E \), then there is a singularity of \( f \) in \( \text{Int } \gamma \).
Due W, Apr 11.

53. Prove Dulac’s Criterion:
Suppose \( E \subset \mathbb{R}^2 \) is a simply connected open set, \( f : E \to \mathbb{R}^2 \) and \( B : E \to \mathbb{R} \) are \( C^r \), \( r \geq 1 \), and \( \text{div}(Bf) \) is not identically zero on \( E \) and is of one sign on \( E \). Then no closed orbit of the system \( \dot{x} = f(x) \) lies wholly within \( E \). (The function \( B \) is termed a “Dulac function.” Note that it can change sign in \( E \).)
Hint. Resist the urge to apply the Product Rule.
Due W, Apr 18.
54. Show that the planar system
\[ \dot{x} = 2xy, \quad \dot{y} = 2xy - x^2 + y^2 + 1 \]
has no closed orbits.
Hint. Figure out why no closed orbit can intersect the \( y \)-axis. Show that a Dulac function is \( B(x, y) = 2/x^2 \). Due W, Apr 25.

55. Show that the planar system
\[ \dot{x} = x(1 + x^2 - 2y^2), \quad \dot{y} = -y(1 - 4x^2 + 3y^2) \]
has no closed orbits by showing that there exist positive constants \( r \) and \( s \) such that \( B(x, y) = x^{-r}y^{-s} \) is a Dulac function. Due W, Apr 25.

56. It can be shown (by constructing a suitable Poincaré Annular Region) that the van der Pol oscillator
\[ \dot{x} = y, \quad \dot{y} = -x + \lambda(1 - x^2)y \]
(studied by Balthasar van der Pol in the 1920's) has at least one limit cycle for all \( \lambda > 0 \). Use the Dulac function \( B(x, y) = (x^2 + y^2 - 1)^{-\frac{1}{2}} \) (discovered by Leonid Cherkas in the late 20th century) to show that the cycle is unique. Hint. The function \( B \) fails to exist on the unit circle. Due W, Apr 25.

57. 312/4 (only Figures 4 and 5) and 313/7. You may use the Bendixson Index Theorem to check, but compute the index using the definition and a labelled drawing showing your computation, as was done with the first two examples done in class. Due M, Apr 30.

58. Resolve the isolated singularity of
\[ \dot{x} = 3x^2 + xy + y^3, \quad \dot{y} = 5xy + 2y^2 - 7x^2y + 4xy^3 \]
at \((0,0)\) by a single polar blow-up.
Hint. It is computationally simpler to examine only the homogeneous quadratic part and see if it is determining. Due M, Apr 30.

59. Let \( U \) be an open interval in \( \mathbb{R} \) containing 0. Suppose \( f : U \to \mathbb{R} \) is \( C^1 \) and that \( f(0) = 0 \).
   a. Show that the function \( g \) defined by
   \[ g(x) = \begin{cases} \frac{f(x)}{x} & \text{if } x \neq 0 \\ f'(0) & \text{if } x = 0 \end{cases} \]
is continuous on \( U \).
   b. Show that \( f(x) = xg(x) \) for all \( x \in U \), so that we have “factored” \( x \) out of \( f(x) \). Due M, Apr 30.
60. Define the transformation \( \Xi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \setminus \{(x,y) : y = 0 \text{ and } x \neq 0\} \) for blowing up an isolated singularity at \((0,0)\) in the \(x\)-direction. Use it to derive the formulas
\[
\dot{\eta} = \frac{1}{v}[p(v\eta, v) - \eta q(v\eta, v)] \quad \text{(defined at } 0 \text{ by the limit)}
\]
\[
\dot{v} = q(v\eta, v).
\]
Due M, Apr 30.

61. Resolve the singularity at \((0,0)\) of the system
\[
\begin{align*}
\dot{x} &= x + 3y + x^3 + 3x^2y - y^3 \\
\dot{y} &= y(-x + x^2 + 3xy).
\end{align*}
\]
Due M, Apr 30.