1 Solution of Homework I

Problem 1.1. As far as two-dimensional geometry is concerned, Hilbert’s Proposition 1 reduces to one simple statement: any two different lines either intersect in one point, or are parallel. Give three or more further useful formulations of this statement.

Answer. Here are three possible answers:

- Any two different lines which are not parallel, have a unique point of intersection.
- If two lines have two or more points in common, they are equal.
- Two different lines have at most one point in common.

Lemma 1 (Proclus’ Lemma). In any affine plane,

- a third line intersecting one of two parallel lines intersects the other one, too.
- a third line parallel to one of two parallel lines, is parallel to the other one, too.

Problem 1.2. Explain why Proclus’ Lemma is an easy consequence of the uniqueness of parallels. Convince yourself that, conversely, Proclus’ Lemma implies the uniqueness of parallels.

Answer. Suppose towards a contradiction that the transversal \( t \) intersects one of the parallel lines \( l \) and \( m \), but not the other one.

We may assume that \( P \) is the intersection point of lines \( t \) and \( m \). If lines \( t \) and \( l \) would not intersect, then \( t \) and \( m \) would be two different parallels of line \( l \) through point \( P \). This contradicts the uniqueness of parallels.

Conversely, we now assume Proclus’ Lemma to be true and check the uniqueness of the parallel to a given line \( l \) through a given point \( P \) not on line \( l \). Let \( m \) and \( t \) be two parallels to line \( l \) through point \( P \)—these may be equal or different lines. The line \( t \) is a transversal intersecting one of the two parallel lines \( m \parallel l \) at point \( P \). Hence it intersects the second line \( l \), too, contrary to the assumption. The only possibility left is that \( m = t \). Hence the parallel to a given line through a given point is unique.
Problem 1.3. Prove from the axioms of incidence (I.1) (I.2) (I.3) that there exist two different lines through every point.

Answer. By axiom (I.3b), there exist at least three points that do not lie on a line. We call them A, B and C. Let any point P be given. In the case that point P is different from all three points A, B, C, we draw the three lines PA, PB and PC. At least two of them are different since A, B, C do not lie on a line. In the case that point P is one of the three points A, B, C, we draw the three lines AB, BC and CA. These are three different lines, and two of them go through the given point P. In both cases we have obtained two different lines through the arbitrary point P.

Definition 1 (Projective plane). A projective plane is a class of points, and a class of lines satisfying the axioms:

P.1 Every two points lie on exactly one line.

P.2 Every two lines intersect in exactly one point.

P.3 There exist four points of which no three lie on a line.

Problem 1.4. Prove from the axioms (P.1) (P.2) (P.3) that there exist three different lines through every point.

Convince yourself that on every line of a projective plane lie at least three points.

Answer. By axiom (P.3), there exist at least four points A, B, C, D of which no three lie on a line. Let any point P be given. In the case that point P is different from all four points A, B, C, D, we draw the lines PA, PB, PC and PD. At least three of them are different since no three points among A, B, C, D lie on a line. In the case that point P is one of the four points A, B, C, D, we get three different lines through the given point P among the six lines connecting A, B, C, D. In both cases we have obtained three different lines through the arbitrary point P.

The proof confirming that three points lie on every line is done quite similarly, interchanging the roles of points and lines.

Definition 2 (Isomorphism of incidence planes). Two incidence planes are called isomorphic if and only if there exists a bijection between the points of the two planes, and a bijection between the lines of the two planes such that incidence is preserved.

Problem 1.5. Given two incidence geometries, it is not obvious whether they are isomorphic. By corresponding labelling of the points in both geometries, show an isomorphism between the two six-point incidence geometries in the figure on page 3.

Answer.
Problem 1.6. Give a highly symmetric illustration for the Fano plane based on an equilateral triangle. Your symmetric illustration is really isomorphic to the projective plane from page 4 above, which was obtained above by completion. (See the lecture or the online lecture notes)

Denote the seven points with the same names in both drawings, consistently in a way to show the isomorphism. After you have obtained the isomorphism, color the lines with seven different colors. Give the corresponding lines in the other model the same colors.

Answer. The figure on page 4 give an illustration based on an equilateral triangle. To check that this symmetric illustration is isomorphic to the illustration I have given on the left side, one needs to names the points in both illustrations in a way that the incidence relations hold for the same names. Thus the isomorphism is given by the correspondence of names.

To find such an isomorphism, the key observation is that a triangle can be mapped...
Figure 3: Fano’s seven-point projective plane

Figure 4: The symmetric drawing of the Fano-plane is really isomorphic to the projective completion of the affine plane of order 2.

to any triangle, but afterwards the correspondence of the remaining points is uniquely determined.
Problem 1.7 (Scheduling problem I). Make a 6 day schedule for a school with 25 students. Each day the students are divided in a different way into 5 groups of 5 students. Never are two students in the same group more than one time during the week.

Answer. We use the coordinate plane $\mathbb{Z}_5 \times \mathbb{Z}_5$. With addition and multiplication modulo 5, the set $\mathbb{Z}_5$ is a field, because 5 is a prime number. Hence the coordinate plane $\mathbb{Z}_5 \times \mathbb{Z}_5$ is an affine plane with order 5. As explained in the lecture notes each line has 5 of points in a affine plane of order $n = 5$. The lines can be partitioned into $n + 1 = 6$ classes, each containing 5 parallel lines.

Hence one set of parallel lines determines the groups of students for one weekday. For each of the 6 weekdays, we use parallel lines with a different slope 0, 1, 2, 3, 4, and finally on one day the vertical parallel lines.

One can make a picture (see page 5) of the schedule by drawing the $5 \times 5$ pattern of dots separately for every day. The five parallel lines in every pattern are indicated by different symbols for their points. One needs curved lines to connect all five points of a line, which I did not do. Clearly such a picture contains more insight than a bare-bone list.

Problem 1.8 (Scheduling problem II). Make a 5 day schedule for a school with 15 students. Each day the students are divided in a different way into 5 groups of 3 students. Never are two students in the same group more than one time during the week.
Answer. We cut down the solution of the last problem and retain just the 15 students from three groups on the sixth weekday. For the remaining five days, we obtain five groups with three students, as required.
Problem 1.9. Show that every finite incidence geometry has at least as many lines as points.

http://en.wikipedia.org/wiki/De_Bruijn%E2%80%93Erd%C5%91s_theorem_%28incidence_geometry%29

Problem 1.10. Count the points of the projective space $\text{PG}(n,q)$.

Answer. The $n$-dimensional projective space over the Galois field $GF(q)$ has $\frac{q^{n+1}-1}{q-1}$ points since the Galois field has $q$ points.

In general, the number $k$-dimensional subspaces of $\text{PG}(n,q)$ is given by the product

$$\left(\begin{array}{c} n+1 \\ k+1 \end{array}\right)_q = \prod_{i=0}^{k} \frac{q^{n+1-i} - 1}{q^{i+1} - 1} = \frac{(q^{n+1} - 1)(q^n - 1)\cdots(q^{n+1-k} - 1)}{(q-1)(q^2 - 1)\cdots(q^{k+1} - 1)}$$

This is a Gaussian binomial coefficient, a $q$-analogue of a binomial coefficient. In the limit $q \to 1$, one gets back to the ordinary binomial coefficient.

Main Theorem 1 (Veblen-Young Theorem). Every finite projective space of geometric dimension $n \geq 3$ is isomorphic with a $n$-dimensional projective space over some finite Galois field $GF(q)$, one of the $\text{PG}(n,q)$.

Remark 1. Note that this result holds only for dimension at least 3. In sharp contrast, the projective planes are much harder to classify, as not all of them are isomorphic with one of the $\text{PG}(2,q)$. The Desarguesian planes are those which are isomorphic with a $\text{PG}(2,q)$. These are exactly those planes satisfying Desargues’s theorem. But there exist non-Desarguesian planes—and even the prime power conjecture about their order is still open.
The smallest 3-dimensional projective space is built over the field $GF(2) = \mathbb{Z}_2$ and is denoted by $PG(3, 2)$. It has 15 points, 35 lines, and 15 planes. Each of the 15 planes contains 7 points and 7 lines. As geometries, these planes are isomorphic to the Fano plane. Every point of $PG(3, 2)$ is contained in 7 lines and every line contains three points. In addition, two distinct points are contained in exactly one line and two planes intersect in exactly one line. In 1892, Gino Fano was the first to consider such a finite geometry - a three dimensional geometry containing 15 points, 35 lines, and 15 planes, with each plane containing 7 points and 7 lines.

Figure 6: $PG(3, 2)$ but not all the lines are drawn.

Problem 1.11 (Kirkman’s Schoolgirl Problem 1850). Kirkman’s schoolgirl problem is a problem in combinatorics proposed by Rev. Thomas Penyngton Kirkman in 1850 as Query VI in The Lady’s and Gentleman’s Diary (pg.48). The problem states:

Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast.

Answer. There are 7 days of the week, and 3 girls in each of 5 group. There are $35 = 7 \times 5$ different combinations for three girls to walk together.

Two of the seven non-isomorphic solutions of Kirkman’s schoolgirl problem provide a visual representation of the Fano 3-space $PG(3, 2)$. Some diagrams for this problem can be found at:

Beutelspacher, Albrecht; Rosenbaum, Ute (1998), Projective geometry: from foundations to applications, Cambridge University Press,

Each color represents the day of the week (seven colors, blue, green, yellow, purple, red, black, and orange). The definition of a Fano space says that each line is on three
points. The figure represents this showing that there are 3 points for every line. This is the basis for the answer to the schoolgirl problem. This figure is then rotated 7 times. There are 5 different lines for each day, multiplied by 7 (days) and the result is 35. Then, there are 15 points, and there are also 7 starting lines on each point. This then gives a representation of the Fano 3-space, PG(3,2).

http://mathworld.wolfram.com/KirkmansSchoolgirlProblem.html

gives the solution

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http://en.wikipedia.org/wiki/Kirkman%27s_schoolgirl_problem#Notes

http://en.wikipedia.org/wiki/Thomas_Kirkman

http://en.wikipedia.org/wiki/Finite_geometry

**Remark 2.** The problem can be generalized to \( n \) girls, where \( n \) must be an odd multiple of 3 thus \( n \equiv 3 \mod 6 \), walking in triplets for \( (n - 1)/2 \) days, with the requirement, again, that no pair of girls walk in the same row twice. It is this generalization of the problem that Kirkman discussed first, while the famous special case \( n = 15 \) was only proposed later. A complete solution to the general case was given by D. K. Ray-Chaudhuri and R. M. Wilson [15] in 1968, but had already been settled by Lu Jiaxi [16] in 1965.