2 Solution of Test II

[20] Problem 2.1. Given is a quadrilateral with two pairs of congruent opposite sides. Prove in neutral geometry that the two pairs of opposite sides are parallel. Provide a drawing.

Answer. Given is the quadrilateral $\square ACBD$ with two pairs of congruent opposite sides $AC \cong BD$ and $BC \cong AD$. We draw the diagonal $AB$ and get the congruent triangles $\triangle ABC \cong \triangle BAD$, as one confirms by SSS congruence. Hence the angles $\beta = \angle ABC$ and $\alpha = \angle BAD$ are congruent. We see that the diagonal $AB$ transverses the lines of the opposite sides $DA$ and $BC$ with congruent $z$-angles. By Euclid I.27, we conclude that the opposite sides are parallel.
Problem 2.2. Given is a triangle $\triangle ABC$. Prove that $a < b$ implies $\alpha < \beta$.

Answer. In $\triangle ABC$, we assume for sides $a = BC < AC = b$. We have to compare the angles $\alpha = \angle CAB$ and $\beta = \angle ABC$ across these two sides.

We transfer the shorter side $BC$ at the common vertex $C$ onto the longer side. Thus one gets a segment $CD \cong BC$, with point $D$ between $C$ and $A$. Because the $\triangle BCD$ is isosceles, it has two congruent base angles

$$\delta = \angle CDB \cong \angle DBC$$

Because $C \ast D \ast A$, we get by angle comparison at vertex $B$

$$\delta = \angle DBC < \beta = \angle ABC$$

Now we use the exterior angle theorem for $\triangle ABD$. Hence

$$\alpha = \angle CAB < \delta = \angle CDB$$

By transitivity, these three equations together imply that $\alpha < \beta$. Hence the angle $\alpha$ across the smaller side $CB$ is smaller than the angle $\beta$ lying across the greater side $AC$. In short, we have shown that $a < b \Rightarrow \alpha < \beta$. 
Problem 2.3. Given is any angle $\angle BAC$. Construct the angular bisector.

Construction 1. We choose congruent segments $AB \cong AC$ on the sides of the given angle. Draw the line $BC$, and transfer the base angle $\angle ABC$ to the ray $\overrightarrow{BC}$, on the side of line $BC$ opposite to vertex $A$. On the new ray, we transfer segment $AB$ to get the new segment $BD \cong BA$. The ray $\overrightarrow{AD}$ is the bisector of the given angle $\angle BAC$.

Figure 3: The angular bisector

Question. Reformulate the description of this construction precisely, and as short as possible.

Answer. One transfers two congruent segments $AB$ and $AC$ onto the two sides of the angle, both starting from the vertex $A$ of the angle. The perpendicular, dropped from the vertex $A$ onto the segment $BC$, is the angular bisector.

Proof of validity. By assumption, the three points $A, B, C$ do not lie on a line. By construction, points $A$ and $D$ lie on different sides of line $BC$. Hence the segment $AD$ intersects line $BC$, say at point $M$. Steps (1) and (4) confirm the congruence of three triangles.

Step (1): We confirm that $\triangle AMB \cong \triangle DMB$.

Answer. The matching pieces used for the proof are stressed in the figure on page 4. Indeed, by construction, $\angle ABC \cong \angle DBC$. Hence $\angle ABM \cong \angle DBM$. (It does not matter whether $M$ lies on the ray $\overrightarrow{BC}$ or the opposite ray.) Too, we have a pair of congruent adjacent sides: Indeed $BD \cong BA$ by construction, and $BM \cong BM$. Now SAS congruence implies $\triangle AMB \cong \triangle DMB$.

Step (2): Explain carefully why $\angle AMB$ is a right angle.

Answer. Because of $\triangle AMB \cong \triangle DMB$, we get $\angle AMB \cong \angle DMB$. Because point $M$ lies between $A$ and $D$, these are two supplementary angles. Hence they are right angles.

Step (3): Prove that $M$ lies between points $B$ and $C$. 
Answer. By construction $AB \cong AC$ and hence the triangle $\triangle ABC$ is isosceles. It has congruent base angles $\beta = \angle ABC \cong ACB$, and as a consequence of the exterior angle theorem, we know that they are acute.

We can now rule out that $M = B$ since we would obtain an isosceles triangle $\triangle AMC$ with a right base angle. Too, we can rule rule out that $M \ast C \ast B$. In this case the triangle $\triangle AMC$ would have a right angle and an obtuse angle $\angle MCA$ supplementary to the acute base angle $\beta$. The impossible case is illustrated in the figure on page 4. Similarly, we see that that neither the cases $M = C$ nor $M \ast B \ast C$ are possible. Hence

\[ B \ast M \ast C. \]

Step (4): Finally, confirm that $\triangle DMB \cong \triangle AMC$.

Answer. Again the pieces used for the proof are stressed in the figure on page 5. One uses SAA congruence. The two triangles have the congruent vertical angles $\angle DMB \cong$
∠AMC. By construction we get the congruent sides $DB \cong AC$ and finally the congruent angles $∠DBM = ∠DBC \cong ∠ACB = ∠ACM$ since the triangle $\triangle ABC$ is isosceles.

**Conclusion** : From the triangle congruences in step (1) and (4), we conclude that $\triangle AMB \cong \triangle AMC$. Hence $∠MAB \cong ∠MAC$, and $MB \cong MC$. ¹ Hence ray $\overrightarrow{AM} = \overrightarrow{AD}$ lies inside the given angle $∠BAC$, which is bisected.

¹Since point $M$ lies on the line $BC$, the last congruence shows that $M$ lies between $B$ and $C$, too.
Problem 2.4 (Construction with classical tools in neutral geometry).

Give a construction of an equilateral triangle, and an angle of 60° with compass and straightedge in neutral geometry.

(a) Do the construction and give a description.

(b) What can one say about the angles at the vertices of the triangle, in neutral geometry? Why are extra steps needed for the construction of the 60° angle?

(c) At which vertex can you get the angle of 60° even in neutral geometry, nevertheless?

(d) Convince yourself once more that all reasoning has been done in neutral geometry. Additional to Hilbert’s axioms, which intersection property do you need.

Figure 7: The construction of a 60° angle at the center $O$ is possible with straightedge compass in neutral geometry.

Answer. (a) Description of the construction. Draw any segment $AB$. The circles about $A$ through $B$, and about $B$ through $A$ intersect in two points $C$ and $D$. We get an equilateral triangle $\triangle ABC$. Now we draw a third circle about $C$ through point $A$, which passes through point $B$, too. With the two circles drawn earlier, the third circle has additional intersection points $H$ and $F$. Finally, we draw the
segments $AF$, $BH$ and $CD$. These are the perpendicular bisectors of the sides of triangle $\triangle ABC$. All three intersect in one point $O$ inside triangle $\triangle ABC$. At vertex $O$, one gets six congruent angles which add up to $360^\circ$, hence they are all $60^\circ$.

(b) **The angles at the vertices may not be $60^\circ$.** Extra steps are needed, because, in neutral geometry, the angle sum of a triangle may be less than two right angles. All we can conclude about the angles at the vertices $A, B, C$, is their congruence. This follows because the angles opposite to congruent sides are congruent. Still they may all three be less than $60^\circ$.

(c) **The $60^\circ$ angles appear at the center.** At vertex $O$, one gets six congruent angles which add to $360^\circ$, hence they are all $60^\circ$.

(d) **The circle-circle intersection property is needed.** All justifications can be given in neutral geometry. Besides Hilbert’s axioms, we only need the circle-circle intersection property.