4 Homework

Definition 1 (Reflection across a Line). Given is a line \( l \). The reflection across line \( l \) is the mapping \( r : P \mapsto P' \) which maps an arbitrary point \( P \) to a point \( P' \) meeting the following requirements:

- If \( P \) lies on the symmetry axis \( l \), we put \( P' := P \).
- If \( P \) does not lie on the symmetry axis, the point \( P' \) is specified by requiring

(i) \( P \) and \( P' \) lie on different sides of \( l \).

(ii) the lines \( l \) and \( PP' \) are perpendicular, intersecting at \( F \).

(iii) \( PF \cong P'F \).

Problem 4.1 (Reflection of a point across a line). Given is a line \( AB \) and a point \( P \) not on this line. We have to construct the reflected point \( P' \), using only Hilbert tools.

- Describe the construction.

- Provide a (large enough) figure, with named objects.
Problem 4.2 (Justification for the construction of the reflection across a line). Given is a line $AB$ and a point $P$ not on this line. We have constructed the reflected point $P'$, using only Hilbert tools. Justify the construction, for the case occurring in your drawing.
Problem 4.3. Given is a line \( l \) which is a plain mirror, a bundled light source at point \( P \), and a point \( Q \) on the same side of the mirror, to which one wants the light ray to be reflected.

Give a construction of the light ray from \( P \) to \( Q \), reflected by the mirror. Provide a drawing and an explanation.
Problem 4.4. In many technical and physical applications—for example for sensitive measurement of electrical current—one uses a small mirror attached to a twisting thin wire. A light beam shines onto the mirror and its reflected beam is depicted on a scale, which can be rather far away.

Give the reason why turning the mirror by an angle $\theta$ results in turning the reflected beam by angle $2\theta$. Provide a drawing.
The remaining problems refer to Euclidean geometry. Explanations about the quadratrix are given on the last pages.

**Problem 4.5.** In Cartesian \((x, y)\) coordinates, we put the unit circle with center \(O = (0, 0)\) through the point \(A = (0, 1)\).

- Use straightedge and compass to construct a regular 12-gon inscribed into the unit circle, with point \(A\) as one vertex.
- For all twelve vertices of your polygon, give exact root expressions of their Cartesian coordinates.
Problem 4.6. In the figure on page 7, several points of the quadratrix have been constructed by means of a regular 24-gon. In the first quadrant, one obtains on the quadratrix the points $C_1$ through $C_6 = B$.

- **Find exact root expressions of the Cartesian coordinates of $C_2, C_3, C_4, C_6$ and mark the points,—in the figure drawn in red,—by their coordinates. (This part can be done with high school knowledge.)**

- **Find exact root expression for Cartesian coordinates of $C_1, C_5$ and mark the two points,—in the figure drawn in red,—by their coordinates. You may wish to look at [http://math2.uncc.edu/~frothe/numalg.pdf](http://math2.uncc.edu/~frothe/numalg.pdf) p. 48 of subsection 3.8 Arithmetic of the Complex Numbers, Roots of unity. There is given the justification of the table below.**

<table>
<thead>
<tr>
<th>$k$</th>
<th>$15^\circ$</th>
<th>$30^\circ$</th>
<th>$45^\circ$</th>
<th>$60^\circ$</th>
<th>$75^\circ$</th>
<th>$90^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos k15^\circ$</td>
<td>$\frac{\sqrt{3}+\sqrt{6}}{4}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{6}-\sqrt{2}}{4}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\sin k15^\circ$</td>
<td>$\frac{\sqrt{3}}{4}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{6}+\sqrt{2}}{4}$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
Figure 1: Construction of 6 points on the quadratrix by means of a regular 24-gon.
Problem 4.7. Do the construction of points on the quadratrix for a regular 12-gon, in an exact drawing. In the first quadrant including the positive y-axis, one obtains 3 exact points on the quadratrix.
Problem 4.8. Do the construction of points on the quadratrix for a regular 32-gon,—in an exact drawing. In the first quadrant including the positive y-axis, one obtains 8 exact points on the quadratrix.
4.1 The Quadratrix

The quadratrix of Hippias is generated in the following way. In Cartesian \((x, y)\) coordinates, we take two objects:

- a horizontal line \(y = \frac{\pi \theta}{2}\);
- a turning ray \(\frac{y}{x} = \tan \theta\).

intersecting at a point \((x, y)\). We may imagine that the parameter \(t = \frac{\pi \theta}{2}\) as the time. With increasing time, the horizontal line \(y = t\) is moving up while the ray is rotating counterclockwise. Their intersection points yield the quadratrix. Classically, Hippias considered the curve for \(0 \leq t \leq 1\). Obviously, we can extend the parametric equations to all real parameter \(t\).

The motion of the two objects is coordinated in such a way that for time \(t = 0\) the process is started with the \(x\)-axis and a ray from the origin pointing to the right. At time \(t = 1\), we obtain the horizontal line \(y = 1\) and a ray along the positive \(y\)-axis. Hence their intersection point \((x, y) = (0, 1)\) lies on the quadratrix. With the real parameter \(t\), the parametric equations for the quadratrix are

\[
x = t \cot \frac{\pi t}{2}
\]

\[
y = t
\]

The angle \(\theta = \frac{\pi t}{2}\) has to be measured in radians.

**Remark.** On the graphing calculator, it is easier to obtain the reflected curve. With \(x\) and \(y\) switched, we get the equation

\[
y = x \cot \frac{\pi x}{2}
\]

which can easily be graphed.

We now ask, which points of the quadratrix are constructible with compass and straightedge. Suppose that the regular polygon with \(N\) sides is constructible. Hence the polygon with \(q = 4N\) sides is constructible, too. For the integers \(q \geq 3\) and \(p \neq 0\), the vertices of the regular \(q\)-gon lying in the first quadrant are those on the rays with polar angles \(\theta = \frac{p \pi}{2q}\) for \(p = 1, 2, \ldots, q\). By dividing the vertical segment from \((0, 0)\) to \((1, 0)\) into \(q\) congruent parts, we construct the corresponding horizontal lines. Thus for the time parameters \(t = \frac{p}{q}\) with \(p = 1, 2, \ldots, q\), the horizontal line and the ray have both been constructed. As intersection points, we get on the quadratrix the constructible points

\[
x = \frac{p \cos \frac{p \pi}{2q}}{q \sin \frac{p \pi}{2q}}
\]

\[
y = \frac{p}{q}
\]
with $p = 1, 2, \ldots, q$. We do this construction successively for the regular polygons with 8, 16, 32, \ldots sides, and obtain a dense set of constructible points on the quadratrix.

**Question.** Is the point were the quadratrix intersect the $x$-axis constructible?

**Answer.** In this case, the above construction fails, because the turning ray points to the right, and lies entire on the corresponding horizontal line $y = 0$. No intersection point is obtained. Nevertheless, with a bid of calculus, we can obtain the point $(x_0, 0)$ of the quadratrix on the $x$-axis. Indeed, by means of limits we get

\[
x_0 = \lim_{t \to 0} t \cot \frac{\pi t}{2} = \lim_{t \to 0} \frac{t \cos \pi t/2}{\sin \pi t/2} = \lim_{t \to 0} \frac{t}{\sin \pi t/2} = \frac{2}{\pi}
\]

Since Lindemann’s work, it is known that $\pi$ is transcendental, hence we conclude that the point of the quadratrix on the $x$-axis is not constructible.

**Proposition 1.** All points $(x, y)$ of the quadratrix, for which both coordinates are algebraic have rational coordinate $y \neq 0$. Indeed, if $y \neq 0$ is rational, then $x$ is always algebraic.

**Proposition 2.** The only points on the quadratrix (4.1) which are constructible with compass and straightedge are given by

\[
x = \frac{p \cos \frac{\pi p}{2q}}{q \sin \frac{\pi p}{2q}}
\]

\[
y = \frac{p}{q}
\]

with either $q = 1, 2$ and $p$ odd, or $q \geq 3$, $p \neq 0$. In the latter case the fraction $\frac{p}{q}$ has to be in lowest terms and a regular polygon with $q$ sides has to be constructible.