5 Solution of Homework

Construction 1 (Construct the midpoint of a segment with Hilbert tools). Given is a segment $AB$. Choose any point $C$ not on the line $AB$. Transfer angle $\angle CAB$ into the half-plane of $AB$ opposite to point $C$, with the second endpoint $B$ as vertex and ray $\overrightarrow{BA}$ as one side. Next transfer congruent segments $AC \cong BD$ onto the newly produced legs of these two angles. Finally, line $\overline{AB}$ and segment $CD$ intersect at the midpoint $M$.

![Diagram of Hilbert's construction of the midpoint]

**Figure 1:** Hilbert’s construction of the midpoint

Problem 5.1. Complete the following justification of the above construction 1.

Question. Why does point $M$ exist?
Question. Why does point $M$ lie between $A$ and $B$. More specifically, why is it impossible that $M \ast A \ast B$? Here is a sketchy drawing for that impossibility $M \ast A \ast B$.  

1 Apply the exterior angle theorem twice to get a contradiction.

Answer. In triangle $\triangle ABD$, the interior angle at vertex $B$ is $\beta = \angle DBM$, which is smaller than the exterior angle at vertex $\epsilon = \angle BMC$. In triangle $\triangle AMC$, the angle $\epsilon$ from above is interior angle at vertex $M$, and hence smaller than the exterior angle $\alpha = \angle BAC$ at vertex $A$.

Now transitivity yields $\beta < \epsilon < \alpha$. On the other hand, the angles $\angle DBA = \beta$ and $\alpha = \angle CAB$ are congruent by construction. This contradiction rules out the case $M \ast A \ast B$.

\footnote{The drawing, too, occurs in the millenium edition of "Grundlagen der Geometrie", page 26.}
**Question.** Now we know that $M$ lies between $A$ and $B$, we finally can prove that $M$ is the midpoint. Which congruence theorem is used for which triangles? Complete the proof and provide the drawing.

**Answer.** One uses SAA congruence for $\triangle AMC$ and $\triangle BMD$. Indeed, the angles at $A$ and $B$ are congruent by construction, and the angles at vertex $M$ are congruent vertical angles. (Both statements are only true because $M$ lies between $A$ and $B$!) The sides $AC$ and $BD$ opposite to those angles are congruent by construction. Hence the two triangles are congruent, and especially $AM \cong MB$
Problem 5.2. As shown in the figure on page 4, given is a triangle $\triangle ABC$, a point $D$ on the side $AB$, and a ray $r$ emanating from point $D$ into the interior of the triangle. We assume that this ray forms the angle $\epsilon \leq \beta$ with the ray $\overrightarrow{DA}$.

Use the exterior angle theorem and the crossbar theorem to show, in neutral geometry, that the ray intersects the side $AC$. 
Figure 5: Since $\epsilon \leq \beta < \delta$ the ray $r$ intersects the side $AC$.

Answer. We draw the segment $DC$ and use the exterior angle theorem for triangle $\triangle DBC$. In this triangle the angle $\delta = \angle(\overrightarrow{DA}, r)$ is an exterior angle and $\beta = \angle ABC$ is a nonadjacent interior angle. One concludes $\beta < \delta$. Hence the assumptions imply $\epsilon \leq \beta < \delta$, which means that the ray $r$ lies in the interior of the angle $\delta$. We use the crossbar theorem for the triangle $\triangle ADC$, and conclude that the ray $r$ intersects the opposite side $AC$, as to be shown.
Problem 5.3. Given is a line \( l \) which is a plain mirror, a bundled light source at point \( P \), and a point \( Q \) on the same side of the mirror, to which one wants the light ray to be reflected.

Give a construction of the light ray from \( P \) to \( Q \), reflected by the mirror. Provide a drawing and an explanation.

Figure 6: Reflect a light ray from \( P \) to a given point \( Q \).

Answer. We construct the reflected point \( Q' \) of the point \( Q \) to which the light ray is to be directed. The light has to be directed along the segment \( PQ' \). The intersection of this segment with the (line of the) mirror gives the point \( R \) where the light ray is reflected. The reflected light goes from from point \( R \) to the given target point \( Q \).

Proof of validity. Let \( G \) be the foot-point of the perpendicular dropped from point \( Q \) onto the mirror. The angles \( \angle QRG \cong \angle GRQ' \) are congruent. The angle of the incident light ray with the mirror is a vertical angle of \( \angle GRQ' \), and the angle of the reflected light ray with the mirror is \( \angle QRG \). Hence they are congruent, as required for the reflection of a light ray.
Problem 5.4. In many technical and physical applications—for example for sensitive measurement of electrical current—one uses a small mirror attached to a twisting thin wire. A light beam shines onto the mirror and its reflected beam is depicted on a scale, which can be rather far away. Additionally it is assumed that the incoming light ray, and the perpendicular lines to the mirror in both positions lie in a plane. Thus we deal with the two-dimensional problem.

Give the reason why turning the mirror by an angle $\theta$ results in turning the reflected beam by angle $2\theta$. Provide a drawing.

Answer. The problem is easiest under the assumption that the light ray hints the mirror exact at its point of rotation (respectively in three dimensions on its axis of rotation). By the law of reflection of light, the angle between the perpendicular to the mirror and the reflected ray is congruent to the angle between the incoming light ray and the perpendicular. Hence $\alpha = \angle LOP_1 \cong \angle P_1OR_1$ in the figure on page 7. The angle between the incoming light ray and its reflected ray is the sum $\angle LOR_1 = 2\alpha$ of these two angles. As the mirror is turning by the angle $\theta$, its perpendicular is turning by the congruent angle $\theta = \angle P_1OP_2$ between the perpendiculars; shown in blue. The angle between the incoming light ray and the perpendicular to the mirror in its second position is $\angle LOP_2 = \alpha + \theta$, with plus sign for the situation as drawn. The angle between the perpendicular and the reflected ray increases by the same angle $\theta$.

After the turn of the mirror has occurred, the angle between the incoming light ray and its reflected ray is $\angle LOR_2 = 2(\alpha + \theta)$. The reflected light ray has been turned by the difference angle $\angle R_1OR_2 = \angle LOR_2 - \angle LOR_1 = 2\theta$.
The remaining problems refer to Euclidean geometry.

**Figure 8**: Construction of 6 points on the quadratrix by means of a regular 24-gon.

**Problem 5.5.** Do the construction of points on the quadratrix for a regular 32-gon,—in an exact drawing constructed with compass and straightedge. In the first quadrant including the positive y-axis, one obtains 8 exact points on the quadratrix. One can use the more historic setup, as in the figure on page 8. Provide your drawing on an extra page to be attached.
Remark. To work in a coordinate system, it is more convenient to put the quadratrix into a position different from the historical one, with $x$ and $y$ from above switched. This has been done in the figure on page 9 and page 10. Take the points $A = (-1, 0), O = (0, 0), B = (1, 0)$. Beginning with the ray $O\hat{A}$ and the vertical line through point $A$, and ending with the ray $O\hat{B}$ and the vertical line through point $B$, we imagine that with increasing time the vertical line is evenly shifting to the left and a ray turning clockwise. The points where the line and the ray intersect are the points of the quadratrix. One checks that the quadratrix in this position has the equation

\begin{equation}
    y = x \cot \frac{\pi x}{2}
\end{equation}

which can conveniently be graphed. Since the function (5.1) is even, in this more convenient position, the $y$-axis gets the axis of symmetry.

\textbf{Figure 9:} The trisection of an angle of $60^\circ$ by means of the quadratrix.

\textbf{Problem 5.6.} In the figure on page 9, the quadratrix is used in a setup with a vertical line moving shifting parallel to the right, and a ray turning clockwise. One obtains the equation $y = x \cot \frac{\pi x}{2}$, convenient to graph the quadratrix.

The figure shows the quadratrix, and how it can be used to trisect the angle of $60^\circ$. Explain the steps of the construction.
Problem 5.7. In the figure on page 10, the quadratrix with equation \( y = x \cot \frac{\pi x}{2} \) is shown once more. It is now used to partition an arbitrary angle into five congruent parts. We assume that the quadratrix is given at the beginning of the construction. Explain the steps how an arbitrary angle is subdivided into five congruent parts.

Answer. The angle to be partitioned is put with vertex \( O \). Its two sides intersect the quadratrix at points \( Q_0 \) and \( Q_5 \). From the intersection points we drop the perpendiculars \( p \) and \( q \) onto the x-axis \( AB \),—where the quadratrix intersects the unit circle. The segment between the foot points \( P \) and \( Q \) is divided into five congruent parts \( PP_1, P_1P_2, P_2P_3, P_3P_4, P_4Q \). The standard Euclidean construction for this step is shown in the figure on page 10, too. We draw four further parallels to \( p \) and \( q \) through the partition points \( P_1, P_2, P_3, P_4 \) and find their intersection points \( Q_1, Q_2, Q_3, Q_4 \) with quadratrix. The rays \( \overrightarrow{OQ_1}, \overrightarrow{OQ_2}, \overrightarrow{OQ_3}, \overrightarrow{OQ_4} \) partition the given angle \( \angle Q_0OQ_5 \) into five congruent parts.
5.1 The Quadratrix

The quadratrix of Hippias is generated in the following way. In Cartesian \((x, y)\) coordinates, we take two objects:

- a horizontal line \(y = \frac{\pi \theta}{2}\);
- a turning ray with \(x = r \cos \theta, y = r \sin \theta\) and \(r \geq 0\).

We may imagine the parameter \(t = \frac{\pi \theta}{2}\) as time. With increasing time, the horizontal line \(y = t\) is moving up while the ray is rotating counterclockwise. Their intersection points \((x, y)\) yield the quadratrix.

The motion of the two objects is coordinated in such a way that for time \(t = 0\) the process is started with the \(x\)-axis and a ray from the origin pointing to the right. At time \(t = 1\), we obtain the horizontal line \(y = 1\) and a ray along the positive \(y\)-axis. Hence their intersection point \((0, 1)\) lies on the quadratrix. With the real parameter \(t\), the parametric equations for the quadratrix are

\[
\begin{align*}
x &= t \cot \frac{\pi t}{2} \\
y &= t
\end{align*}
\]

The angle \(\theta = \frac{\pi t}{2}\) has to be measured in radians. Obviously, we can extend the parametric equations to all real parameter \(t\). We see that the \(x\)-axis turns out to be the axis of symmetry.

**Remark.** The invention of the quadratrix is credited to Hippias of Elis (born around 460 B.C.). He considered the curve only for \(0 \leq t \leq 1\). Hippias used his curve for trisection of an angle. Dinostratus (ca. 350 B.C.) is usually credited with applying it for squaring of the circle. These arguments are assuming that the quadratrix is available as an entire curve at the beginning of the constructions. With this assumption, the arguments are mathematically rigorous.

We need to make clear that Hippias’ constructions cannot be achieved with straightedge and compass alone. Neither the trisection of an angle of 60°, nor squaring of the circle can be achieved by compass and straightedge. Indeed, it can be shown that the only points on the quadratrix (5.2) which are constructible with compass and straightedge are those obtained via the constructible regular polygons.