11 The Regular Pentagon

11.1 The Euclidean construction with the Golden Ratio

The figure on page 561 shows an easy Euclidean construction of a regular pentagon. The justification of the construction begins by first considering the regular 10-gon. Furthermore—as customary since Legendre—we take a shortcut by solving a quadratic equation.

By a "slice of a regular \( n \)-gon", I mean an isosceles triangle with base of which is one side of the polygon, and the third vertex of which is the center of the circum circle.

Problem 11.1. Find two equiangular triangles within the slice of the regular \( 10 \)-gon. Set up the relevant proportion to determine the side length \( x \) of the 10-gon within a circum circle of radius 2. Get a quadratic equation for \( x \) and solve it.

Figure 11.1: Two similar triangles appear within the slice of the regular 10-gon.

*Similar triangles appear within the slice of the regular 10-gon.* Let \( O \) be the center of the circum circle and \( AB \) a side of the 10-gon. The isosceles triangle \( \triangle OAB \) has base angles 72°. The top angle is 36°—which is just half of the base angles. Hence it is natural to bisect one of them. Let \( BC \) be the bisector. The triangles \( \triangle OAB \) and \( \triangle BAC \) are...
equiangular. By Euclid VI.4, the sides of equiangular triangles are proportional. Hence we get a proportion

\[(11.1) \quad \frac{|AB|}{|AO|} = \frac{|AC|}{|AB|}\]

Both triangles have a pair of congruent base angles. By Euclid I.9, they are isosceles, and hence $OC \cong BC \cong BA$. By assumption we set $|AO| = 2$ and $|AB| = x$, and hence get $|AC| = |AO| - |OC| = 2 - x$. Thus the proportion (11.1) yields

\[
\frac{x}{2} = \frac{2 - x}{x}
\]

\[x^2 = 2(2 - x)\]

The quadratic equation for $x$ has the two solutions $x_{1,2} = -1 \pm \sqrt{5}$.

**Definition 11.1 (Golden ratio).** The ratio of the longer to the shorter side of the isosceles triangle with angles $36^\circ$, $72^\circ$, $72^\circ$ is called the **golden ratio**. Its value is

\[
\phi = \frac{\sqrt{5} + 1}{2} = 1.61803\ldots
\]

**Problem 11.2.** Check that $\phi = \frac{2}{x}$ and $1 - \phi = -\frac{x}{2}$, and that these are the two solutions of the quadratic equation $\phi^2 - \phi - 1 = 0$.

**Problem 11.3.** Do and describe, and justify the Euclidean construction of the regular pentagon. For simplicity, choose the radius of the circum circle equal to 2 units.

**Answer.** One begins the construction with a circle around $O$ of radius $|OA| = 2$. Next one bisects the segment $OO'$ perpendicular to $OA$. Let $D$ be the midpoint.

The right triangle $\triangle OAD$ has legs $|OA| = 2$, $|OD| = 1$ and hence hypothenuse $|AD| = \sqrt{5}$, by the Theorem of Pythagoras. The circle around $D$ through point $O$ intersects the hypothenuse in point $E$. Segment subtraction yields $|AB| = |AE| = \sqrt{5} - 1 = x$, which is just the side of the regular 10-gon.

Finally, a circle around $A$ through point $E$ intersects the circum circle drawn in the beginning in two adjacent vertices $B$ and $B_5$ of a pentagon. Finally, one gets all vertices $B_1 B_2 B_3 B_4 B_5$ of the regular pentagon since its sides are congruent, and its vertices all lie on the circum circle. The construction is done in the figure on page 561.

**Problem 11.4.** Alternatively, the equation for the side $x$ of the 10-gon can be justified with Euclid III.36, the theorem about secants and tangents to a circle. This is done in the figure on page 562. Explain which circle you use, and how to get a tangent and a pair of secants from point $B$ outside this circle.
**Justification using secants in a circle.** Let $AF$ and $BC$ be the angular bisectors in triangle $\Delta OAB$. We use the circle through points $O, C$ and $F$. The center $H$ of this circle lies on the third angular bisector of the angle with vertex $O$.

Since $\angle OCH \cong \angle COH = 18^\circ$ and $\angle ACB \cong \angle CAB = 72^\circ$, angle subtraction at vertex $C$ confirms that the angle $\angle BCH = 180^\circ - 18^\circ - 72^\circ = 90^\circ$ is a right angle.

We use the circle through points $O, C$ and $F$. The tangent from point $B$ to this circle touches at point $C$. Finally, we use Euclid III.36—about secants and tangents to a circle—and get

\[ |BC|^2 = |BO| \cdot |BF| \]
\[ x^2 = 2(2 - x) \]

as above.

\[ \square \]

### 11.2 Relation between the sides of pentagon and 10-gon

**Problem 11.5.** Let $M$ be the midpoint of the rhombus $\square ABCB_5$. Use the Theorem of Pythagoras for the right triangle $\Delta OMB$ and calculate the side $y = |BB_5|$ of the pentagon.

The length of a side for the pentagon. The right triangle $\Delta OMB$ has the legs $|OM| = \frac{2 + x}{2}$ and $|MB| = \frac{y}{2}$. The hypotenuse is $|OB| = 2$. Hence the Theorem of Pythagoras
Figure 11.3: Euclid’s theorem of secants gives another derivation of the golden mean.

yields
\[
\frac{(2 + x)^2}{4} + \frac{y^2}{4} = 4
\]
from which we get
\[
y^2 = 16 - (2 + x)^2
\]
We can use \(x = \sqrt{5} - 1\) and get the side of the pentagon explicitly:
\[
y^2 = 16 - (2 + x)^2 = 16 - (\sqrt{5} + 1)^2 = 10 - 2\sqrt{5}
\]
\[
y = \sqrt{10 - 2\sqrt{5}}
\]

Theorem 11.1 (A Relation between the Sides of Pentagon and 10-gon (Euclid XIII.10)). The sides of the regular pentagon, hexagon, and 10-gon inscribed into the same circle are the sides of a right triangle.

Problem 11.6. Use the information gathered so far to give a short algebraic proof of Euclid XIII.10.
Algebraic proof of Euclid XIII.10. Since $x^2 = 4 - 2x$ we get
\[ y^2 = 16 - 4 - 4x - x^2 = 12 + (2x^2 - 8) - x^2 = 4 + x^2 \]
By the converse Pythagorean Theorem, $y$ is the hypothenuse of a right triangle with legs 2 and $x$, as claimed by Euclid XIII.10.

Figure 11.4: Proving Euclid XIII.10.

A more traditional proof of Euclid XIII.10. In the regular pentagon $BCDEF$, bisecting the angle $\angle FOB$ produces the vertex $A$ of a regular 10-gon with the same circum circle around $O$. We drop the perpendicular from $O$ onto the side $AB$ of the 10-gon, and let $N$ be its intersection point with the side $FB$ of the pentagon. Euclid’s theorem of tangent and secants (Euclid III.36) gives rectangles for the squares of segments $FO$ and $BA$.

Question. Why is the ray $\overrightarrow{FO}$ a tangent to the circle through points $O, N$ and $B$ and touches at point $O$. What does Euclid III.36 imply.

Answer. The angles $\angle FON \cong \angle FBO = 54^\circ$ are congruent. Hence they are angle between tangent and secant and circumference angle for the same chord $ON$. Hence we conclude that
\[ |FO|^2 = |FN| \cdot |FB| \]
Of course, you get the same result from the observation that the triangles $\triangle FON \sim \triangle FBO$ are equiangular and hence similar.

**Question.** Why is the ray $\overrightarrow{BA}$ a tangent to the circle through points $A, N$ and $F$ and touches at point $A$. What does Euclid III.36 imply.

**Answer.** The angles $\angle BFA \cong \angle BAN = 18^\circ$ are congruent. Hence they are circumference angle and angle between tangent and secant for the same chord $NA$. Hence we conclude that

$$|BA|^2 = |BN| \cdot |BF|$$

Of course, you get the same result because the triangles $\triangle BFA \sim \triangle BAN$ are equiangular and hence similar. Adding the two equations results in

$$|FO|^2 + |BA|^2 = |FN| \cdot |FB| + |BN| \cdot |BF| = (|FN| + |BN|) \cdot |BF| = |BF|^2$$

Indeed, as to be shown, the sum of the squares of the hexagon side $|FO|$ and the 10-gon side $|BA|$ is the square of the pentagon side $|BF|$.

By a "10-gon three-side-diagonal", I mean a diagonal spanning over three adjacent sides of the 10-gon, hence corresponding to a central angle three times the central angle of a side.

**Theorem 11.2 (A Relation between the diagonals of pentagon and the 10-gon).**

The diagonal of the regular pentagon is the hypothenuse of a right triangle with the side of the hexagon, and the three-side-diagonal of the 10-gon as legs.

**Problem 11.7.** Use the figure on page 565 to prove Theorem 11.2.

**Solution of Problem 11.7.** In the regular pentagon $BCDEF$, bisecting the angle $\angle FOB$ produces the vertex $A$ of a regular 10-gon with the same circum circle around $O$. We drop the perpendicular from $O$ onto the side $AF$ of the 10-gon, and let $N$ be its intersection point with the diagonal $FC$ of the pentagon. Euclid’s theorem of tangent and secants (Euclid III.36) gives rectangles for the squares of segments $FO$ and $CA$.

**Question.** Why is the ray $\overrightarrow{FO}$ a tangent to the circle through points $O, N$ and $C$ and touches at point $O$. What does Euclid III.36 imply.

**Answer.** The angles $\angle FON \cong \angle FCO = 18^\circ$ are congruent. Hence they are angle between tangent and secant and circumference angle for the same chord $ON$. Hence we conclude that

$$|FO|^2 = |FN| \cdot |FC|$$

Of course, you get the same result from the observation that the triangles $\triangle FON \sim \triangle FCO$ are equiangular and hence similar.

\[\text{We substitute } AB \mapsto AF, FB \mapsto FC.\]
Question. Why is the ray $\overrightarrow{CA}$ a tangent to the circle through points $A, N$ and $F$ and touches at point $A$. What does Euclid III.36 imply.

Answer. The angles $\angle CFA \cong \angle CAN = 54^\circ$ are congruent. Hence they are circumference angle and angle between tangent and secant for the same chord $NA$. Hence we conclude that

$$|CA|^2 = |CN| \cdot |CF|$$

Of course, you get the same result because the triangles $\triangle CFA \sim \triangle CAN$ are equiangular and hence similar. Adding the two equations results in

$$|FO|^2 + |CA|^2 = |FN| \cdot |FC| + |CN| \cdot |CF| = (|FN| + |CN|) \cdot |CF| = |CF|^2$$

Indeed, as to be shown, the sum of the squares of the hexagon side $|FO|$ and the 10-gon three-diagonal $|CA|$ is the square of the pentagon diagonal $|CF|$.

Problem 11.8. We use the notation as in the figure on page 565. Check that the side
and three-side diagonal of the 10-gon, and the side and diagonal of the pentagon satisfy

\[
\frac{|AF|}{|OA|} = \frac{\sqrt{5} - 1}{2} \quad \text{and} \quad \frac{|AC|}{|OA|} = \frac{\sqrt{5} + 1}{2}
\]

\[
\frac{|FB|}{|OA|} = \frac{\sqrt{5} - \sqrt{5}}{2} \quad \text{and} \quad \frac{|FC|}{|OA|} = \frac{\sqrt{5} + \sqrt{5}}{2}
\]

Give a more simple algebraic proof of Theorem 11.2.

11.3 The construction with Hilbert tools

Problem 11.9. Find the irreducible monic polynomial with zero \(\frac{\sqrt{5} - 1}{2}\). Find all algebraic conjugates, and show that the number is an algebraic integer.

Problem 11.10. Find the irreducible polynomial with zero \(z_1 = \sqrt{\frac{5 - \sqrt{5}}{2}}\). Find all its algebraic conjugates, and show that the number is a totally real algebraic integer.

Solution of Problem 17.4. Simple arithmetic shows that \((2z^2 - 5)^2 - 5 = 0\) and hence

\[P(z) := z^4 - 5z^2 + 5 = 0\]

is the monic polynomial in the ring \(\mathbb{Z}[z]\) with zero \(z_1\). Hence \(z_1\) is an algebraic integer. The Eisenstein criterium tells:

For a polynomial to be irreducible in the ring \(\mathbb{Z}[z]\), it is sufficient that all its coefficients except the leading one are divisible by the same prime number \(p\), but the constant coefficient is not divisible by \(p^2\).

The polynomial \(P(z)\) is irreducible since it satisfies this condition for \(p = 5\). Hence its zeros are exactly the algebraic conjugates of \(z\). Obviously, these zeros are

\[
\sqrt{\frac{5 + \sqrt{5}}{2}}, \quad -\sqrt{\frac{5 + \sqrt{5}}{2}}, \quad \sqrt{\frac{5 - \sqrt{5}}{2}}, \quad -\sqrt{\frac{5 - \sqrt{5}}{2}}
\]

which all four turn out to be real.

The set of all lengths constructible by Hilbert tools is called the Hilbert field and denoted by \(\Omega\). By Proposition 17.13, all lengths constructible by Hilbert tools are totally real. Hence the domain \(T\) of all totally real Euclidean numbers satisfies \(\Omega \subseteq T \subseteq K\), where \(K\) denotes the Euclidean field. \(^{47}\)

To come back to the pentagon, we have seen in Problem 17.4 by means of algebraic arguments that \(z_1 \in T\). From Artin’s Theorem, we conclude that even \(z_1 \in \Omega\).

Here is an even easier direct proof leading to the same conclusion: By Proposition 14.1, the Hilbert field \(\Omega\) is the smallest field with properties

\(^{47}\)By a theorem of Emil Artin, the Hilbert field \(\Omega\) is exactly the field \(T\) of the totally real algebraic Euclidian numbers.
(1) $1 \in \Omega$.

(2) If $a, b \in \Omega$, then $a + b, a - b, ab \in \Omega$.

(2a) If $a, b \in \Omega$ and $b \neq 0$, then $\frac{a}{b} \in \Omega$.

(3) If $a, b \in \Omega$, then $\sqrt{a^2 + b^2} \in \Omega$.

For the side and diagonal of the pentagon, we get

$$\sqrt{\frac{5 \pm \sqrt{5}}{2}} = \sqrt{\left(\frac{1 \pm \sqrt{5}}{2}\right)^2 + 1}$$

which we can now see to be obtained recursively by the steps (1)(2)(3) specified above.

Thus we conclude that its side and its diagonal, and hence the pentagon is constructible with Hilbert tools. It is now time to really do a construction by Hilbert tools!

**Problem 11.11.** Use the three right triangles from above, and find a construction of the regular pentagon by Hilbert tools. For simplicity, I have chosen 4 as radius of the circum circle.

Construction of the pentagon by Hilbert tools. We can use the three right triangles which came up above:

(i) Triangle $\triangle ODA$ has legs $|OA| = 2, |OD| = 1$ and hypothenuse $|AD| = \sqrt{5}$.

(ii) Euclid XIII.10 yields a right triangle with legs $2, x$ and hypotenuse $y$.

(iii) Triangle $\triangle OMB$ has legs $|OM| = \frac{2+x}{2}, |MB| = \frac{y}{2}$, and hypothenuse $|OM| = 2$.

We begin with a segment $|OA| = 2$ and erect the perpendiculars at its endpoints $O$ and $A$. Transfer a segment $|OD| = 1$ onto the first perpendicular, as well as to the hypothenuse of the resulting triangle $\triangle ODA$.

We get a segment $|ED| = \sqrt{5} - 1 = x$. and transfer this segment onto the second perpendicular and the ray opposite $\overrightarrow{AO}$ to get segments $|AE| = |AF| = x$. The hypothenuse of the right triangle $\triangle OAE$ is $|OE| = y$, congruent to the side of the pentagon in a circle of radius 2.

We transfer this segment onto the first perpendicular on both sides of $O$, and get segments $|OG| = |OH| = y$.

The isosceles triangle $\triangle FGH$ is one of the slices of a pentagon with circum circle around $F$ with radius $|FG| = |FH| = 4$. The four remaining slices are obtained by angle transfer at center $F$. 

\[\square\]
Problem 11.12. Again, we assume that $|OA| = 2$. Check that the construction done in the figure on page 569 yields the segments

\[ |OC| = \frac{\sqrt{5} - 1}{2} \quad \text{and} \quad |OB| = \frac{\sqrt{5} + 1}{2} \]

\[ |DC| = \frac{5 - \sqrt{5}}{2} \quad \text{and} \quad |DB| = \frac{5 + \sqrt{5}}{2} \]

Problem 11.13. Use the information gathered so far justify the construction of the regular pentagon with Hilbert tools—as done in the figure on page 569.

Validity of the construction. Here are the major steps of the construction done in the figure on page 569:

(i) On the radius $OA$, a perpendicular is erected. Its midpoint is $D$.

(ii) Construction the inner and outer angular bisectors $\overrightarrow{DC}$ and $\overrightarrow{DB}$ of the angle $\angle ODA$. Let $C$ and $B$ be the intersection points of the bisectors with the line $OA$.

(iii) Transfer segment $DC$ at point $B$ onto the perpendicular erected on line $OA$. On both sides of this line, one gets the congruent segments $BE \cong BF \cong DC$.  

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Figure 11.7: Still another construction of the pentagon with Hilbert tools.

(iv) Transfer segment $DB$ at point $C$ onto the perpendicular erected on line $OA$. On both sides of this line, one gets the congruent segments $CG \cong CH \cong DB$.

(v) Finally connects the vertices $AHFEG$ of a regular pentagon.

The side and diagonal of the pentagon were obtained in Problem 11.8. By Problem 11.12, they agree with the segments we have just constructed.
11.4 Variants of the Euclidean construction

Problem 11.14 (Another Euclidean construction of the regular pentagon). From the construction done in the figure on page 570, and \(|OA| = 2\) given, calculate exact expressions for the segments

(i) \(|OC|\) and \(|OD|\).

(ii) \(|BC|\) and \(|BD|\).

Answer. We follow the steps of the construction to get exact expressions for the segments.

(i) \(|OC| = \sqrt{5} - 1\) and \(|OD| = \sqrt{5} + 1\). From \(|OB| = 2\) and \(|MB|^2 = |MO|^2 + |OB|^2 = 5\) one gets \(|MC| = |MD| = |MB| = \sqrt{5}\) and

\[
|OC| = |MC| - |MO| = \sqrt{5} - 1
\]
\[
|OD| = |MD| + |MO| = \sqrt{5} + 1
\]

(ii) Two more applications of the Theorem of Pythagoras:

\[
|BC|^2 = |BO|^2 + |OC|^2 = 2^2 + (\sqrt{5} - 1)^2 = 10 - 2\sqrt{5}
\]
\[
|BD|^2 = |BO|^2 + |OD|^2 = 2^2 + (\sqrt{5} + 1)^2 = 10 + 2\sqrt{5}
\]

\[
|BC| = \sqrt{10 - 2\sqrt{5}} \quad \text{and} \quad |BD| = \sqrt{10 + 2\sqrt{5}}
\]
By comparison with the results of Problems ?? and ??, we get a proof that the construction does yield a regular pentagon.

**Problem 11.15.** *Get a formula for \( \sin 18^\circ \). Guess and check the exact values of \( \sin k\phi \), \( \cos k\phi \) and \( \tan k\phi \) for multiples of \( \phi = 18^\circ \) and \( k = 1, 2, 3, 4 \). This is just an exercise in clever use of the calculator!*

*Answer. We can use the figure on page 559. The slice of the regular 10-gon is the isosceles triangle \( \triangle OAB \) with top angle 36°. We drop the perpendicular from center \( O \) onto the side \( AB \). The foot point \( G \) is its midpoint. The right triangle \( \triangle OGA \) yields*

\[
\sin 18^\circ = \frac{|GA|}{|OA|} = \frac{\sqrt{5} - 1}{4}
\]

As a further hint, we use the expressions obtained in Problem 11.14:

\[
\frac{|OC|}{|OB|} = \frac{\sqrt{5} - 1}{2} \quad \text{and} \quad \frac{|OD|}{|OB|} = \frac{\sqrt{5} + 1}{2}
\]

\[
\frac{|BC|}{|OB|} = \frac{\sqrt{5} - \sqrt{5}}{2} \quad \text{and} \quad \frac{|BD|}{|OB|} = \frac{\sqrt{5} + \sqrt{5}}{2}
\]

But some of them are larger than one, so you still need to divide. Here is the result:

<table>
<thead>
<tr>
<th>( k\phi )</th>
<th>( \sin k\phi )</th>
<th>( \cos k\phi )</th>
<th>( \tan k\phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>18°</td>
<td>( \frac{\sqrt{5} - 1}{4} )</td>
<td>( \frac{\sqrt{5} + \sqrt{5}}{8} )</td>
<td>( \frac{\sqrt{2}}{20} ) ( 5 - \sqrt{5} )^{3/2}</td>
</tr>
<tr>
<td>36°</td>
<td>( \sqrt{\frac{5 - \sqrt{5}}{8}} )</td>
<td>( \frac{\sqrt{5} + 1}{4} )</td>
<td>( \frac{\sqrt{10}}{20} ) ( 5 - \sqrt{5} )^{3/2}</td>
</tr>
<tr>
<td>54°</td>
<td>( \sqrt{\frac{5 + \sqrt{5}}{4}} )</td>
<td>( \sqrt{\frac{5 - \sqrt{5}}{8}} )</td>
<td>( \frac{\sqrt{2}}{20} ) ( 5 + \sqrt{5} )^{3/2}</td>
</tr>
<tr>
<td>72°</td>
<td>( \sqrt{\frac{5 + \sqrt{5}}{8}} )</td>
<td>( \sqrt{\frac{5 - 1}{4}} )</td>
<td>( \frac{\sqrt{10}}{20} ) ( 5 + \sqrt{5} )^{3/2}</td>
</tr>
<tr>
<td>90°</td>
<td>1</td>
<td>0</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

**11.5 A false pentagon**

**Problem 11.16** *(A false pentagon construction). Into the construction done in the figure on page 572, put a Cartesian coordinate system with \( O = (0, 0) \), \( A = (2, 0) \) and \( B = (0, 2) \).*

(i) Calculate exact expressions for the coordinates of \( B_1 \).

(ii) Calculate exact expressions for the coordinates of \( B_2 \).

(iii) Determine exactly the ratio \( \frac{|B_3B_2|}{|B_2B_1|} \)
Figure 11.9: This construction does not yield a regular pentagon.

Check of the false pentagon construction. We need to calculate the coordinates for intersections points of circles.

(i) The coordinates of $B_1$ are $\left(\frac{\sqrt{55}}{4}, \frac{3}{4}\right)$. Point $B_1$ is an intersection point of the circles $C_{irc_1}$ and $C_{irc_2}$. Since $|OB| = 2$ and $|BM|^2 = |BO|^2 + |OM|^2 = 5$, they have the equations:

- $C_{irc_1}: x^2 + y^2 = 4$
- $C_{irc_2}: x^2 + (y - 2)^2 = 5$

Subtraction yields

$$(y - 2)^2 - y^2 = 1,$$ and hence $y = \frac{3}{4}$.

With the equation for $C_{irc_1}$ we get

$$x^2 + \frac{9}{16} = 4,$$ and hence $x = \frac{\sqrt{55}}{4}$.

(ii) The coordinates of $B_2$ are $\left(\frac{3\sqrt{55}}{16}, -\frac{23}{16}\right)$. Point $B_2$ is an intersection point of
the circles $Circ_1$ and $Circ_3$. They have the equations:

\[ Circ_1 : \quad x^2 + y^2 = 4 \]
\[ Circ_3 : \quad \left( x - \frac{\sqrt{55}}{4} \right)^2 + \left( y - \frac{3}{4} \right)^2 = 5 \]

Subtraction yields

\[
\left( x - \frac{\sqrt{55}}{4} \right)^2 - x^2 + \left( y - \frac{3}{4} \right)^2 - y^2 = 1 \\
\frac{-\sqrt{55}}{2}x + \frac{55}{16} - \frac{3}{2}y + \frac{9}{16} = 1 \\
\sqrt{55}x + 3y = 6
\]

Plugging into the equation for $Circ_1$ yields

\[
x^2 + \left( 2 - \frac{\sqrt{55}}{3}x \right)^2 = 4 \\
\frac{64x^2}{9} - \frac{4\sqrt{55}x}{3} = 0 \\
x_1 = 0 \quad \text{and} \quad x_2 = \frac{3\sqrt{55}}{16}
\]

The solution $x = 0$ leads back to point $B$. The new intersection point $B_2$ corresponds to the solution $x_2$. Plugging into the linear relation of $x$ and $y$ yields

\[
\sqrt{55}x + 3y = 6 \\
\sqrt{55} \times \frac{3\sqrt{55}}{16} + 3y = 6 \\
y = 2 - \frac{55}{16} = \frac{-23}{16}
\]

(iii) The exact ratio of the longer to the shorter side is

\[
\frac{|B_3 B_2|}{|B_2 B_1|} = \frac{\frac{3\sqrt{55}}{8}}{\sqrt{5}} = \frac{3\sqrt{11}}{8}
\]