18 The Lunes of Hippocrates

18.1 The historic lunes

Definition 18.1 (Lune). A lune consists of two circular arcs having a common chord and lying on the same side of this chord. The interior of the lune is the crescent shaped area formed by the difference of the interiors of the corresponding circles. It is bounded by the lune’s two arcs.

Hippocrates of Chios (ca. 430 B.C.) posed the problem:

(Hippocrates Problem). Find the lunes which are constructible and squarable with straightedge and compass.

He gave three examples of constructible lunes. They are obtained beginning with the following two assumptions:

(a) The two circular sectors corresponding to the lune’s arcs have the same area.

(b) The central angles of the two circular arcs are commensurable.

Hippocrates of Chios is credited with discovering three such lunes; two more were discovered in the 18th century. In the 20th century Tschebatorev and Dorodnov (1947) proved these five are the only ones.

Lemma 18.1. Assumption (a) implies the lune is squarable.
Proof. Let $CD$ be the common chord of the two arcs of the lune. Let $S_A$ be the circular sector corresponding to the lune’s longer arc and have center $A$, and $S_B$ be the circular sector corresponding to the lune’s shorter arc and have center $B$. Both sectors are delimited by the radiuses from their respective center to the endpoints of common chord $CD$. The sector $S_B$ lies on the other side of the lune’s shorter arc and does not intersect its interior area. We add to the lune the latter sector and then subtract the sector $S_A$, and obtain the kite $\square ACBD$. Since by assumption (a) the two sectors have the same area, the lune has the same area as the kite and hence is squarable.

Lemma 18.2. Assumption (a) and (b) together imply

\[
\frac{\alpha}{\beta} = \frac{n}{m} = \frac{b^2}{a^2}
\]

with integers $n, m \geq 1$. Indeed $n < m$ becomes possible only if one or both or angles $\alpha$ and $\beta$ are more than $360^\circ$.

Proof. We now use assumption (b). Let the angle $\varepsilon$ be the (greatest) common measure of the central angles $\alpha$ and $\beta$ of the lune’s longer and shorter arcs around centers $A$ and $B$. Hence $\alpha = n\varepsilon$ and $\beta = m\varepsilon$ with integers $n > m \geq 1$. Subdivision of the sector $S_A$ yields $n$ congruent sectors with center $A$, and similarly, subdivision of the sector $S_B$ yields $m$ congruent sectors with center $B$. All sectors we have obtained have the same central angle $\varepsilon$. Hence they are similar.

By assumption (a), the two circular sectors $S_A$ and $S_A$ have equal area, hence the $n$ small sectors of radius $a$ around center $A$ with central angle $\varepsilon$ have together the same area as $m$ similar sectors with central angle $\varepsilon$ and radius $b$ around center $B$. Since the areas of similar figures are proportional to the square of their linear dimension, we conclude $na^2 = mb^2$, and finally get equation (18.1).

Definition 18.2 (Circular segment). A circular segment is bounded by an arc and a chord. A segment of central angle $\varepsilon$ is obtained from the circular sector with the same angle and arc by subtraction or addition of the triangle with vertices at the endpoints of its circular arc and the center of the circle. For a short arc $0 < \varepsilon < 180^\circ$, the triangle is subtracted, for a long arc $180^\circ < \varepsilon < 360^\circ$, the triangle is added to the sector.

Lemma 18.3. Two circular segments with the congruent central angles and which are both bounded by the long, or both by the short arc are similar. There areas have the same ratio as the squares of their radius.

Lemma 18.4. The areas of similar figures are proportional to the square of their linear dimension.

Proposition 18.1. Assumption (a) and (b) together imply the lune has the same area as a polygon with vertices on its arcs.
Proof. After subdivision of the arcs $\alpha$ and $\beta$, we obtain not only similar circular sectors, but also similar segments with the central angle $\varepsilon$. The endpoints of these segments divide the arc $\alpha$ of the lune into $n$ arcs with central angle $\varepsilon$, and the inner arc $\beta$ of the lune $m$ arcs with central angle $\varepsilon$.

The $n$ circular segments around center $A$ have together the same area as the $m$ similar circular segments around center $B$. This is clear from equation (18.1) and Lemma 18.3.

We add to the lune the $m$ segments on the other side of the lune’s inner arc $\beta$, and subtract the $n$ segments with vertices on the lune’s external arc $\alpha$. We obtain a polygon with vertices on the arcs of the lune which has the same area as the lune. \hfill \Box

For which values of the integers $n$ and $m$ can we construct squarable lunes? Let $\alpha = n\varepsilon$ and $\beta = m\varepsilon$ be the angles of the lune arcs. Hippocrates has found a squarable and constructible lune for the following three cases:

(a) $n = 2, m = 1$
(b) $n = 3, m = 1$
(c) $n = 3, m = 2$

The case (a) is easiest to guess: one puts $\alpha = 180^\circ, \beta = 90^\circ$. The longer arc of the lune is the circum-circle of an isosceles right triangle $\triangle CED$ and hence its center $A$ is the midpoint of the hypothenuse $CD$. The shorter arc has its center $B$ in the forth vertex of square $\square CEDB$. Hence we know that $b/a = \sqrt{2}$ as required.

In the case (b), we need to get $b/a = \sqrt{3}$. On the longer arc $\alpha$ of the lune lie three congruent segments $CE, EF, FD$. We begin by choosing an arbitrary length $l$ for them. We know that the endpoints of the last and first segment are the endpoints of the common chord. since $m = 1$, the segment $CD$ is the only one on the inner arc of the lune. From equation (18.1) and similarity we get the distance $|CD| = \sqrt{3}|CE|$.

The quadrilateral $\square CEFD$ is a symmetrical trapezoid, the sides of which are given. We construct the trapezoid and its circum-circle. Thus we get the longer arc of the lune and its center $A$. Thus we get the central angle $\varepsilon = \angle CAE$. By adding the central angles of the three congruent arcs $\angle CAE, EAF$ and $FAD$, we see that $\angle ELG = \alpha = 3\varepsilon$ is the central angle of the longer arc of the lune. Since $m = 1$ by assumption, we find the center $B$ of the second circle from $\varepsilon = \angle CBD$. Too, we know that the trapezoid has the same area as the lune.

Problem 18.1. Do the construction described above. One begins by drawing a rhombus consisting of two equilateral triangles with a common side, of length one. Then one constructs the trapezoid, its circum-circle, and finally the lune.

Answer. A detailed construction is given in the figure on page 699.
Figure 18.2: The constructible and squarable $3:1$ lune of Hippocrates.
Proposition 18.2 (Hippocrates 430 B.C.). The external arc of Hippocrates’ 3 : 1 lune is greater than a semicircle.

Proof. We drop the perpendiculars from $A$ and $C$ onto line $BD$. Since by construction $|BD| = \sqrt{3} |AB| > |AC|$, the foot points $F$ and $G$ of the perpendiculars lie inside the segment $BD$. Thus we obtain a rectangle and two right triangles, one of which is $\triangle CDF$ with the acute angles $\angle FDC$ and $\angle FCD$.

(i) The angle $\angle ACD$ is the sum of the right angle $\angle ACF$ and angle $\angle FCD$, and hence obtuse.

(ii) The triangle $\triangle ACD$ has the obtuse angle $\gamma$. Hence the Pythagorean comparison implies $|AC|^2 + |CD|^2 < |AD|^2$. By construction, the three segments $BA \cong AC \cong CD$ are congruent, and hence $2|AB|^2 = |AC|^2 + |CD|^2 < |AD|^2$.

(iii) By construction and item (ii)

$$|BD|^2 = 3|BA|^2 = |BA|^2 + 2|BA|^2 < |BA|^2 + |AD|^2$$

(iv) Because of item (iii), the Pythagorean comparison in the triangle $\triangle BAD$, implies that angle $\angle BAD$ is acute.

(v) Since the angle $\angle BAD$ is acute and its vertex lies on the arc $\widehat{BAD}$, Corollary 36 implies that this arc, which is the external arc of Hippocrates’ 3 : 1 lune, is greater than a semicircle.

\[\square\]
Construction 18.1 (Hippocrates construction of a squarable 3 : 2 lune). Draw a circle $C$ with diameter $AB$ around center $K$, and the perpendicular bisector $p$ of radius $KB$. Construct a segment $DH$ such that $|DH|^2/|KB|^2 = 3/2$ and mark a congruent segment $EF \cong DH$ on your ruler. Finally, one places the marked ruler such that the following three requirements are met:

1. The ruler line goes through the point $B$.
2. The point marked $E$ on the ruler lies on the circle $C$.
3. The second point marked $F$ on the ruler lies on the perpendicular bisector $p$.

Let $G$ be the reflection image of $E$ across the perpendicular bisector. We draw a circular arc $EKBG$ with center $L$, and a circular arc $EFG$ with center $I$.

Result: We claim that the lune $\mathcal{L}$ between the two circular arcs, the pentagon $EFGBK$, and the kite $\Box IGLE$ have the same area.

Problem 18.2.

(a) The circle around $L$ contains three congruent circular segments, which are similar to two congruent circular segments of the circle around $I$. Which are these segments.
(b) Prove the claim in (a) by means of circumference angles and the axial symmetry across the perpendicular bisector \( p \).

(c) Prove by means of (b) that the lune \( \mathcal{L} \) has the same area as the pentagon \( \text{EFGBK} \).

(d) Similarly to items (a) and (b), we get three congruent circular sectors of the circle around \( L \), which are similar to two congruent circular sectors of the circle around \( I \). Prove that the first three segments have the same area as the latter two.

(e) Prove by means of (d) that the lune \( \mathcal{L} \) between the two circular arcs has the same area as the kite \( \square \text{IGLE} \).

**Answer.** (a) The circle around \( L \) contains three congruent circular segments since \( |EK| = |KB| \) by construction and \( |EK| = |BG| \) by reflection across the bisector \( p \). Let \( \varepsilon \) be their central angle.

(b) We see that \( \angle ELG = \alpha = 3\varepsilon \) is the central angle of the longer arc of the lune. The circumference angle \( \angle EKG = \varepsilon/2 \) is half of the central angle \( \angle ELK = \varepsilon \).

The neusis used in the construction puts points \( E, F \) and \( B \) on a line. By reflection across the bisector \( p \), we see that the points \( G, F \) and \( K \) are on a line, too. Hence \( \angle EGF = \angle EKF \) is a circumference angle of the inner arc of the lune, too. We get the corresponding central angle \( \angle EIF = \varepsilon \). By reflection across the bisector \( p \), we get \( \angle GIF = \varepsilon \). We see that \( \angle EIG = \beta = 2\varepsilon \) is the central angle of the inner arc of the lune.

(c) We choose a scale such that \( |EK| = |KB| = 1 \) is a unit segment. Hence \( |CD| = |CH| = (\sqrt{3})/2 \) and \( |EF| = |HD| = \sqrt{(3/2)} \). From similar triangles \( \triangle ELK \sim \triangle EIF \) we get the scaling factor

\[
\frac{b}{a} = \frac{|EI|}{|EL|} = \frac{|EF|}{|EK|} = \sqrt{\frac{3}{2}}
\]

\[
\frac{b^2}{a^2} = \frac{3}{2} = \frac{\alpha}{\beta}
\]

confirming the requirement (18.1).

The \( n = 3 \) circular segments around center \( L \) have together the same area as the \( m = 2 \) similar circular segments around center \( I \). We add to the lune the two latter segments \( EF \) and \( FG \) onto the other side of the lune’s inner arc \( \beta \), and cut along the longer arc the three circular segments \( EK, KB \) and \( BG \). We obtain the pentagon \( \text{EFGBK} \) which has the same area as the nearby lune \( \text{EFGBK} \).

(d) Similarly to items (a) and (b), we get three congruent circular sectors of the circle around \( L \), which are similar to two congruent circular sectors of the circle around \( I \). Because of equation (18.3) the first three segments have the same area as the latter two.
This time we add the two circular sectors $EIF$ and $FIG$ to the lune onto the other side of the lune’s inner arc, and cut along the longer arc the three circular sectors $ELK, KLB$ and $BLG$. We obtain the kite $\square IGLE$ which has the same area as the lune.

**Figure 18.5:** Hippocrates $3:2$ lune is smaller than a semicircle.

**Proposition 18.3** (Hippocrates 430 B.C.). *The external arc of Hippocrates’ $3:2$ lune is less than a semicircle.*

**Proof.** By construction $|EF| = \sqrt{3/2} |EK| > |E|$. In triangle $\triangle EFK$ the greater angle lies across the longer side, and hence $\chi = \angle EKF > \angle EFK = \eta$. Since a triangle can have only one obtuse or right angle, we conclude that $\eta$ is acute. Finally the supplementary angle $\angle KFB$ is obtuse.

(i) $2|KF|^2 < |KB|^2$. follows from the Pythagorean comparison for the obtuse isosceles triangle $\triangle KFB$.

(ii) By item (i) and since $EK \cong KB$, and the length $|EF|$ is given by construction

$$|EK|^2 + |KF|^2 < |EK|^2 + \frac{1}{2}|KB|^2 = \frac{3}{2}|KB|^2 = |EF|^2$$

(iii) The angle $\chi = \angle EKF$ is obtuse. by Pythagorean comparison for triangle $\triangle EKF$. 

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(iv) Since the three points \(K, F, G\) lie on a line \(\eta = \angle EKG\) is obtuse by item (iii).
Since the vertex of angle \(\angle EKG\) lies on the arc \(E\widehat{KG}\), Corollary 36 implies that
this arc is less than a semicircle.

\[\square\]

Figure 18.6: Hippocrates squared a suitable union of a circle and a lune.

According to the account of van der Waerden, Hippocrates squared a union of a circle
and a lune in a special case, constructed in the following way. A hexagon is inscribed
into a circle of radius 1, and into a concentric smaller circle \(C\) of radius \(1/\sqrt{6}\) with the
same center \(O\). A circular arc which is tangent at the endpoints \(A\) and \(C\) of two adjacent
sides \(AB\) and \(BC\) of the larger hexagon become the inner arc, and the arc \(A\widehat{BC}\) of the
unit circle the external arc of the lune \(\mathcal{L}\). The two arcs of the lune intersect at 30°.

Proposition 18.4 (Hippocrates). The disjoint union \(\mathcal{L} \cup C\) has the same area as the
union of the triangle \(\triangle ABC\) and the hexagon inscribed into the circle \(C\).

Proof. Let \(S, S', S''\) be three congruent circular segments between the unit circle
and the inscribed hexagon. The first two of them, segments \(S, S'\), have the vertices \(A, B\)
and \(B, C\), respectively. Let \(T\) be the large circular segment with vertices \(A\) and \(C\).

The circular segments \(S\) and \(T\) are similar, since both have arcs intersecting at an
angle of 30°. Since twice the altitude of the equilateral triangle \(\triangle ABO\) is \(|AC| = \)
\[ \sqrt{3} |AB|, \text{ we conclude that} \]

\[
\frac{\text{area}(T)}{\text{area}(S)} = 3
\]

\[ \text{area}(S \cup S' \cup S'') = \text{area}(T) \]

Only two circular segments \( S \) and \( S' \) are cut away from the external arc of the lune \( L \). The third segment \( S'' \) has the same area as the six circular segments chopped away from the small circle \( C \).

The big circular segment \( T \) is added into the inner arc of the lune, to obtain the triangle \( \triangle ABC \). Meanwhile, the smaller hexagon is left from the smaller circle. The disjoint union \( L \cup C \) has the same area as the union of the triangle \( \triangle ABC \) and the hexagon inscribed into the smaller circle \( C \).

\[ \square \]

18.2 Historic remark

B.L. van der Waerden’s book *Science Awakening* [39], contains a detailed account of Hippocrates’ work, as far as it has been reconstructed. The book [25] ”The Ancient Tradition of Geometric Problems” by Wilbur Richard Knorr contains interesting historic information about Hippocrates’ work, too.

Hippocrates of Chios (ca. 430 B.C.) is credited with discovering three squarable lunes; two more were discovered in the 18th century. The manner in which Hippocrates squared his lunes can be learned from a famous fragment by Simplicius (ca. 530 A.D.). According to his own statement, Simplicius had copied word by word from the History of Mathematics of Eudemus (ca. 335 B.C.). Hippocrates’ proofs were preserved through the History of Geometry compiled by Eudemus of Rhodes, which has also not survived, but which was excerpted by Simplicius of Cilicia in his commentary on Aristotle’s Physics. Many scholars of history have attempted to reconstruct this lost work of Eudemus. (Obviously, no such attempt of any historic reconstruction is intended in the present notes.)

Another line of information about Hippocrates of Chios comes from Alexander of Aphrodisias. Alexander was the teacher of Simplicius, the most learned and reliable among the commentators of Aristotle.

Heath concludes that, in proving his result, Hippocrates was also the first to prove that the area of a circle is proportional to the square of its diameter. Hippocrates’ book on geometry with the title ”Elements” in which this result appears, has been lost, but may have formed the model for Euclid’s Elements.

Van der Waerden given the following judgement of Hippocrates:

Looking at all these developments as a whole, we see in the first place that Hippocrates mastered a considerable number of propositions from elementary geometry. The ”Elements of Geometry”, which he has written according to the Catalogue of Proclus, must have contained a large part
of Books III and IV of Euclid, as well as the contents of Books I and II, which were "baby food" for the Pythagoreans.

Hippocrates knows the relation between inscribed angles and arcs, the construction of the regular hexagon; he knows how to circumscribe a circle about a triangle and he knows that a circle can be circumscribed about an isosceles trapezoid. He is familiar with the concept of similarity and he knows that the areas of similar figures are proportional to the squares of homologous sides. He knows not only the Theorem of Pythagoras for the right triangle (Euclid I.47) but also its generalization for obtuse- and acute-angled triangles (Euclid II.12 and II.13). Furthermore, he is able to square an arbitrary rectilinear figure, i.e. to construct a square with the same area. By means of this, he knows how to construct lines whose squares have the ratio 3 : 2 or 6 : 1 to the square on a given line.

Still more important for the evaluation of the mathematical level, reached in Athens during the second half of the fifth century and of Hippocrates in particular, is the excellent demonstrative technique and the high requirements of rigor demanded in the proofs. Hippocrates is not satisfied merely to construct the lunules and to conclude from the drawings that the external boundary is greater than or less than a semicircle; he wants and succeeds to prove this rigorously. One has to remember that the operation with inequalities is a very late achievement of modern mathematics, about which even Euler did not worry much.

18.3 Some historic and less historic exercises

Lemma 18.5. The areas of similar figures are proportional to the square of their homologous sides.

Proposition 18.5. Three similar figures are attached to the sides of a right triangle, with scales proportional to the sides of the triangle. The sum of the areas of the figures put onto the legs of the triangle equals the area of the figure put onto the hypothenuse.

Problem 18.3. Provide drawings with four different examples for this proposition. Convince yourself that the Proposition follows from the Lemma above and the Pythagorean Theorem.

Answer.

Biography of Alhazen, the polymath

Abu Ali al-Hasan ibn al-Haytham
Born: 965 in (possibly) Basra, Persia (now Iraq)
Died: 1040 in (possibly) Cairo, Egypt
Alhazen was born in Basra, in the Iraq province of the Buyid Empire. He probably died in Cairo, Egypt. During the Islamic Golden Age, Basra was a "key beginning of learning", and he was educated there and in Baghdad, the capital of the Abbasid Caliphate, and the focus of the "high point of Islamic civilization". During his time in Buyid Iran, he worked as a civil servant and read many theological and scientific books.

One account of his career has him called to Egypt by Al-Hakim bi-Amr Allah, ruler of the Fatimid Caliphate, to regulate the flooding of the Nile, a task requiring an early attempt at building a dam at the present site of the Aswan Dam. After his field work made him aware of the impracticality of this scheme, and fearing the caliph’s anger, he feigned madness. He was kept under house arrest from 1011 until al-Hakim’s death in 1021. During this time, he wrote his influential Book of Optics. After his house arrest ended, he wrote scores of other treatises on physics, astronomy and mathematics. He later traveled to Islamic Spain. During this period, he had ample time for his scientific pursuits, which included optics, mathematics, physics, medicine, and practical experiments.

http://www-history.mcs.st-andrews.ac.uk/Biographies/Al-Haytham.html

http://en.wikipedia.org/wiki/Alhazen%27s_problem
In elementary geometry, Alhazen attempted to solve the problem of squaring the circle using the area of lunes (crescent shapes), but later gave up on the impossible task. The two lunes formed from a right triangle by erecting a semicircle on each of the triangle’s sides, inward for the hypotenuse and outward for the other two sides, are known as the lunes of Alhazen; they have the same total area as the triangle itself.

**Problem 18.4.** Provide a drawing of the two lunes of Alhazen. Explain and prove how the lunes can be squared.

![Figure 18.8: The lunes of Alhazen can be squared.](image)

**Answer.** Because of the generalization of the Pythagorean Theorem stated as Proposition 18.5, the union $S$ of the semicircles erected onto the legs of the triangle has the same area as the semicircle $T$ erected onto the hypotenuse.

The intersection $S \cap T$ of these figures is the sum of the two circular segments over the legs. We subtract the intersection and conclude that the figures $S \setminus T$, which is the union of the two lunes, and $T \setminus S = \triangle ABC$, which is a triangle, have the same area.

**Problem 18.5.** In the figure on page 709 is shown another lune construction. This time three circular arcs have been used, but two of them fit together with a common tangent. Explain how this lune is squared, and why squaring the lune is possible.

**Construction 18.2 (Another constructible and squarable lune).** Into the semicircle with diameter $AB$, we inscribe the right triangle $\triangle ABC$. Let point $R$ be the midpoint of circular arc $ARB$ opposite to the semicircle $BCA$. We draw the segment
Figure 18.9: Can you square this lune?

RC. The perpendiculars dropped onto the segment RC from vertices A and B have the foot-points Q and P, respectively.

Onto the legs of triangle \( \triangle ABC \) are put quartercircles \( \widehat{BC} \) around P, and \( \widehat{CA} \) around Q. Onto the hypothenuse is put the quartercircle \( \widehat{BA} \) around R. These three quartercircles form a lune \( L \).

Squaring the lune. The circular segments erected onto the three sides of triangle \( \triangle ABC \) are similar since all three are bounded by a quartercircle.

According to Proposition 18.5, the union \( S \) of the circular segments erected onto the legs of the triangle has the same area as the circular segment \( T \) erected onto the hypothenuse. We can obtain the lune \( L \) from the triangle \( \triangle ABC \) by adding the union \( S \) and subtracting circular segment \( T \). Hence the lune \( L \) has the same area as the triangle \( \triangle ABC \).
Another line of information about Hippocrates of Chios comes from Alexander of Aphrodisias. According to Alexander, Hippocrates began with an isosceles right triangle. Two congruent lunes are formed by the semicircle on the legs as external arcs, and the semi-circle circumscribed about the triangle as the inner arc. He proved that the sum of the areas of these two lunes is equal to the area of the triangle.

Problem 18.6. Provide a drawing with named points, and explain the reasoning.

Answer. The reasoning how to square this lune is the same as for the lunes of Alhazen.

Here is how Hippocrates, again according to Alexander, tried to find a second squarable and constructible lune. Take an isosceles trapezoid formed by the diameter of a circle and three consecutive sides of an inscribed regular hexagon.

Problem 18.7. Look at the graph on page 711. Prove that the sum of the areas of a semicircle on a side of the hexagon and the three lunes formed by the semicircles on
Figure 18.11: One can square this simple lune.

Figure 18.12: Can one square the circle?

three sides of the hexagon and by the semicircle circumscribing the trapezoid, is equal to the area of the trapezoid.

Remark. Now there is a tempting speculation: if it were possible to ”square” the three lunes, it would be possible to ”square” the semicircle and hence the circle!
Answer. Since the side of an inscribed hexagon equals the radius of the circle, and the
diameter is double the radius, the union $S$ of four semicircles over sides of the hexagon
has the same area as the big semicircle $T$. We place one of the four semicircles aside.
Thus the intersection $S \cap T$ of the figures consists of the circular segments over three
side of the hexagon, or trapezoid $\Box ABCD$.

We subtract the intersection and conclude that the figures $S \setminus T$, which is the union
of three lunes together with the semicircle put aside, and $T \setminus S = \Box ABCD$, which is
the trapezoid, have the same area.

18.4 The lune equation

Proposition 18.6. Any lune satisfying assumption (a) and (b) satisfies the lune equa-
tion

$$\frac{\sin(n\varepsilon/2)}{\sin(m\varepsilon/2)} = \pm \sqrt{\frac{n}{m}}$$

with integers $n, m \geq 1$.

Proof. The common chord of the two arcs of the lune has the length

$$|CD| = 2a \sin(\alpha/2) = 2b \sin(\beta/2)$$

Since $\alpha = n\varepsilon$ and $\beta = m\varepsilon$ and $na^2 = mb^2$, the equation is easy to confirm.

Problem 18.8. The figure on page 713 is a numerical production of Hippocrates’ 3 : 1
lune, obtained from the lune equation

$$\frac{\sin(3\varepsilon/2)}{\sin(\varepsilon/2)} = \sqrt{3}$$

Convince yourself that the area of the lune equals the area of the kite $\Box ACBD$.

In the figure on page 714 you see the graph for the second solution of the same lune
equation. Convince yourself that in this case, the area of the lune equals the area of the
kite $\Box ACBD$ plus twice the area of the smaller circle around $A$. 

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Figure 18.13: A numerical production of Hippocrates' 3 : 1 lune.

Answer. The figure on page ?? is a numerical production of Hippocrates' 3 : 1 lune. The union $S$ of the three congruent circular sectors $ADD'$, $AD'D''$ and $AD''C$ has the same area as the circular sector $BDC = T$. The lune $L = ADD'D''C$ is obtained from the kite $\square BDAC$ by adding the union $S$ and subtracting the circular sector $T$. Hence the the lune has the same area as the kite $\square BDAC$. 
**Figure 18.14:** The area of the lune equals twice the area of smaller circle plus the area of the kite.

**Answer.** The figure on page 714 is the graph for the second solution of the same lune equation. In order to translate the explanation above to this case, one has to count area positive or negative according to the orientation of their boundary.

The union $S$ of the three congruent circular sectors $ADD'$, $AD'D''$ and $AD''C$ has the area of two small circles $C_A$ around $A$ plus the left sector $smallADC$. The area of $S$ equals the area of the circular sector $BDC = T$. The area of the kite $\square BDAC$ is negative because of the clockwise orientation.

The big lune $L = bigADC - smallADC$ is obtained from the negative kite $\square BDAC$ by adding the union $S$ and subtracting the circular sector $T$.

$$-\text{area}(\square CADB) = -\text{area}(\square CADB) + \text{area}(S) - \text{area}(T)$$
$$= -(\text{area}(\square CADB) + \text{area}(T)) + \text{area}(S)$$
$$= -\text{area}(big ADC) + 2 \times \text{area}(C_A) + \text{area}(small ADC)$$
$$= -\text{area}(L) + 2 \times \text{area}(C_A)$$
$$\text{area}(L) = \text{area}(\square CADB) + 2 \times \text{area}(C_A)$$

Hence the big lune has the same area as the kite $\square CADB$ plus twice the area of the smaller circle.
Remark. If one or both or angles $\alpha = n\varepsilon$ and $\beta = m\varepsilon$ are more than $360^\circ$, we have really not squared the lune as drawn. Instead we have squared combined figure consisting of the lune plus $q = \lfloor n\varepsilon/360^\circ \rfloor$ circles of radius $a$ minus $p = \lfloor m\varepsilon/360^\circ \rfloor$ circles of radius $b$.

We get

$$\lvert CD\rvert = 2a \sin(\alpha'/2) = (-1)^q 2a \sin(\alpha/2) = (-1)^q 2a \sin(n\varepsilon/2)$$

$$\lvert CD\rvert = 2b \sin(\beta'/2) = (-1)^p 2b \sin(\beta/2) = (-1)^p 2b \sin(m\varepsilon/2)$$

where the primed angles are in the range $(0^\circ, 360^\circ)$. The sign in the lune equation turns out to be

$$\frac{\sin(n\varepsilon/2)}{\sin(m\varepsilon/2)} = (-1)^{p+q} \sqrt{\frac{n}{m}}$$

Indeed $n < m$ or the minus sign in the lune equation both become possible, but only in such a situation.

**Problem 18.9.** Solve the lune equation

$$\frac{\sin(n\varepsilon/2)}{\sin(m\varepsilon/2)} = \pm \sqrt{\frac{n}{m}}$$

for $n = 3$, $m = 2$ and $0 < \varepsilon < 180^\circ$.

(a) Set up the quadratic equation and find the exact root expressions for $\cos \varepsilon$.

(b) Find the numerical values for $\varepsilon$ in degrees.

(c) Which values correspond to the historic case of Problem 2, which to Problem 3.

**Answer.**

$$\frac{\sin(3\varepsilon/2)}{\sin \varepsilon} = \frac{\sin(\varepsilon/2) \cos \varepsilon + \cos(\varepsilon/2) \sin \varepsilon}{2 \sin(\varepsilon/2) \cos(\varepsilon/2)} = \frac{\cos \varepsilon}{2 \cos(\varepsilon/2)} + \cos(\varepsilon/2) = \frac{\cos \varepsilon + 2 \cos^2(\varepsilon/2)}{2 \cos(\varepsilon/2)}$$

$$= 2 \cos \varepsilon + 1 \sqrt{2(\cos \varepsilon + 1)}$$

From the lune equation we see that $x = \cos \varepsilon$ satisfies the quadratic equation

$$\frac{(2x + 1)^2}{2(x + 1)} = 3$$

$$2x^2 + x - 2 = 0$$

The solutions are:

$$\cos \varepsilon = \frac{-1 \pm \sqrt{33}}{8}$$
There are two real root, and both correspond to real angles. The numerical values are \( \cos \varepsilon \approx 0.5930703308 \) or \(-0.8430703308\) and \( \varepsilon \approx 53.6^\circ \) or \(147.5^\circ\) in degrees. The lunes are drawn in the figures on page 701 and page 716.

**Problem 18.10.** The figure on page 716 shows another \(3:2\) lune construction.

(a) The circle around \(L\) contains three congruent circular segments, which together turn more than a full circle, and which are similar to two congruent circular segments of the circle around \(I\). Which are these segments.

(b) Prove the claim in (a) by means of circumference angles and the axial symmetry across the perpendicular bisector \(p\).

(c) Similarly to items (a) and (b), we get three congruent circular sectors of the circle around \(L\). Together they are a full circle plus a sector. There are two congruent circular sectors of the circle around \(I\) which are similar to former ones. Prove that the first three segments have the same area as the latter two.
(d) Prove by means of (c) that lune minus circle have the same difference of areas as the kite □IGLE.

Figure 18.16: In this case, the 3 : 2 lune minus the lower circle is squarable.

**Problem 18.11.** Find a lune and a circle in the figure on page 717 with squarable difference of areas. Convince yourself that they have the same difference of areas as the self-crossing pentagon EFGBK, were the area of the upper quadrilateral is counted positive, and the area of the lower triangle is counted negative.

Convince yourself directly that the signed area of the self-crossing pentagon EFGBK, which is really EFGNBKN, and the area of the kite □IGLE are equal.

**Answer.** The lune \( L \) is bounded by the long arc \( \widehat{GFE} \) and the short arc \( \widehat{EMG} \). The lune minus the lower circle \( \mathcal{K} \) around \( L \) is squarable.

Indeed, the two circular segments \( \widehat{FG} \) and \( \widehat{FE} \) inside the upper circle have the same area as the three circular segments \( \widehat{EK}, \widehat{KB}, \) and \( \widehat{BG} \) inside the lower circle.

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All these five segments have congruent central angles $\varepsilon \approx 147^\circ$. We subtract the two first segments from the lune $\mathcal{L}$, and the three latter segments from the circle $\mathcal{K}$. The intersection $\hat{E}\hat{M}\hat{G}\hat{N}$ of the circular segments $\hat{E}\hat{K}$ and $\hat{B}\hat{G}$ appears twice. Thus we get the differences

\[
\mathcal{L} - \left[\hat{F}\hat{G} + \hat{F}\hat{E}\right] = E\hat{M}\hat{G}\hat{F}
\]

\[
\mathcal{K} - \left[\hat{E}\hat{K} + \hat{K}\hat{B} + \hat{B}\hat{G}\right] = \triangle BNK - E\hat{M}\hat{G}\hat{N}
\]

\[
\mathcal{L} - \mathcal{K} = \mathcal{L} - \left[\hat{F}\hat{G} + \hat{F}\hat{E}\right] - \mathcal{K} + \left[\hat{E}\hat{K} + \hat{K}\hat{B} + \hat{B}\hat{G}\right]
\]

\[
= G\hat{M}\hat{E}\hat{F} - \triangle BNK + E\hat{M}\hat{G}\hat{N} = \square G\hat{N}\hat{E}\hat{F} - \triangle BNK
\]

The last difference has the signed area of the self-crossing pentagon $E\hat{F}\hat{G}\hat{B}\hat{K}$.

We can also see directly that this difference equal in area to the kite $\square IG\hat{L}\hat{E}$. Indeed, we subtract the two congruent triangles $\triangle G\hat{F}\hat{I}$ and $\triangle \hat{F}\hat{E}\hat{I}$ inside the upper circle and add the three congruent triangles $\triangle E\hat{K}\hat{L}$, $\triangle \hat{K}\hat{B}\hat{L}$, and $\triangle B\hat{G}\hat{L}$ inside the lower circle. The sum of the areas of the former two is equal the sum of the areas of the latter three triangles.

Subtraction of the former two and addition of the latter three figures from the difference $\square G\hat{N}\hat{E}\hat{F} - \triangle BNK$ yields the kite $\square IG\hat{L}\hat{E}$, which has hence the same area.

### 18.5 Vieta’s and Euler’s lunes

**Problem 18.12.** Solve the lune equation

\[
\frac{\sin(n\varepsilon/2)}{\sin(m\varepsilon/2)} = \sqrt{\frac{n}{m}}
\]

for $n = 4$, $m = 1$ and $0 < \varepsilon < 180^\circ$.

(a) **Set up the cubic equation and find from Cardano’s formula the exact root expression for $2\cos(\varepsilon/2)$**.

(b) **Find the numerical value for $\varepsilon$ in degrees**.

(c) **Draw the lune including its segments and sectors**.

**Answer.**

\[
\frac{\sin(2\varepsilon)}{\sin(\varepsilon/2)} = \frac{2\sin\varepsilon\cos\varepsilon}{\sin(\varepsilon/2)} = \frac{4\sin(\varepsilon/2)\cos(\varepsilon/2)\cos\varepsilon}{\sin(\varepsilon/2)} = 4\cos(\varepsilon/2) \left[2\cos^2(\varepsilon/2) - 1\right]
\]

Hence $x = 2\cos(\varepsilon/2)$ satisfies the cubic equation $x^3 - 2x - 2 = 0$.

From Cardano’s formula we get with $b = -2, c = -2$:

\[
2\cos(\varepsilon/2) = \sqrt[3]{\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}} + \sqrt[3]{-\frac{c}{2} - \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}}
= \sqrt[3]{1 + \frac{19}{27}} + \sqrt[3]{1 - \frac{19}{27}}
\]

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Figure 18.17: The 4:1 lune of Vieta.

There is only one real root. The numerical value is $2\cos(\epsilon/2) \approx 1.765292354$ and $\epsilon \approx 55.58\degree$ in degrees. The lune is drawn in the figure on page 719.

Problem 18.13. Solve the lune equation

$$\sin(n\epsilon/2) \sin(m\epsilon/2) = \pm \sqrt{n/m}$$

for $n = 5$, $m = 1$ and $0 < \epsilon < 180\degree$.

(a) Set up the quadratic equation and find the exact root expressions for $\cos \epsilon$.

(b) Find the numerical values for $\epsilon$ in degrees.

(c) How many real angles occur as solutions.

Answer.

$$\frac{\sin(5\epsilon/2)}{\sin(\epsilon/2)} = 4\cos^2 \epsilon + 2\cos \epsilon - 1$$
From the lune equation we see that \( x = \cos \varepsilon \) satisfies the quadratic equation

\[
4x^2 + 2x - 1 = \pm \sqrt{5}
\]

The solutions are:

\[
\cos \varepsilon = \frac{-1 \pm \sqrt{5 \pm 4\sqrt{5}}}{4}
\]

There are two real roots and two complex roots. Only the root with both plus signs corresponds to a real angle. The numerical value is \( \cos \varepsilon \approx 0.6835507455 \) and \( \varepsilon \approx 46.9^\circ \) in degrees.

**Problem 18.14.** Solve the lune equation

\[
\frac{\sin(n\varepsilon/2)}{\sin(m\varepsilon/2)} = \pm \sqrt{\frac{n}{m}}
\]

for \( n = 5, m = 3 \) and \( 0 < \varepsilon < 180^\circ \).

(a) **Set up the quadratic equation and find the exact root expressions for \( \cos \varepsilon \).**

(b) **Find the numerical values for \( \varepsilon \) in degrees.**

(c) **How many real angles occur as solutions.**

---

**Figure 18.18:** The constructible 5 : 1 lune of Euler.
Figure 18.19: The constructible 5 : 3 lune of Euler.

Answer.

\[
\frac{\sin(5\varepsilon/2)}{\sin 3(\varepsilon/2)} = \frac{4\cos^2\varepsilon + 2\cos\varepsilon - 1}{2\cos\varepsilon + 1}
\]

From the lune equation we see that \(t = \cos\varepsilon\) satisfies the quadratic equation

\[
\sqrt{3}(4x^t + 2t - 1) = \sigma\sqrt{5}(2t + 1)
\]

The solutions are with \(\sigma = \pm 1:\)

\[
\cos\varepsilon = \frac{\sigma\sqrt{5} - \sqrt{3} \pm \sqrt{20 + 2\sigma\sqrt{15}}}{4\sqrt{3}}
\]

For \(\sigma = +1\), there are two real roots, both correspond to a real angle. The numerical values are \(\cos\varepsilon \approx 0.8330386705\) or \(-0.6875414461\) and \(\varepsilon \approx 33.6^\circ\) or 133.4° in degrees.

For \(\sigma = -1\), there are two real roots, only one corresponds to a real angle. The numerical values are \(\cos\varepsilon \approx -0.0674839681\) or \(-1.078013256\) and \(\varepsilon \approx 93.9^\circ\) in degrees.
Remark. Viêta found around 1593 the 4 : 1 lune leading to a cubic equation. The constructible and squarable 5 : 1 and 5 : 3 lunes were found in 1766 by Martin Johan Wallenius. Too, Leonard Euler made the same discovery around 1771, published in "Solutio problematis geometrici circa lunules a circulis formatas". The two lunes from Euler and the two not so obvious ones by Hippocrates are once more obtained algebraically from the lune equation in an article by Th. Clausen [11] published in 1840. The results of Hippocrates were apparently not known at that time. Except for Hippocrates, none of these authors consider a combination of lune and circle to be squared.

Remark. A popular account of some of the material from this section is contained in William Dunham’s book "Journey through Genius". A short introduction and some biographic information is given by


![Image](image.png)

Figure 18.20: A constructible 5 : 3 lune minus circle around A.

18.6 About transcendental numbers

A number \( \alpha \) is called algebraic if there exists a nonzero integer polynomial \( p \in \mathbb{Z}[x] \) such that \( p(\alpha) = 0 \). In that case \( \alpha \) is called a root of the polynomial \( p \).

The set of all real or complex algebraic numbers is denoted by \( \mathbb{A} \). As stated in Theorem 17.1, the set of algebraic numbers is both a countable and a field. Obviously,
all rational numbers are algebraic, but the converse is not true. We know that \( \sqrt{2} \) and 
\( i = \sqrt{-1} \) are two important examples of irrational but algebraic numbers.

The real or complex numbers, which are not algebraic are called **transcendental**.

**Definition 18.3.** The numbers \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are called \( \mathbb{Z} \)-linearly independent if the dependence relation
\[
k_1 \alpha_1 + k_2 \alpha_2 + \cdots + k_n \alpha_n = 0
\]
with integers \( k_1, k_2, \ldots, k_n \) holds only for all \( k_1 = k_2 = \cdots = k_n = 0 \).

The numbers \( \gamma_1, \gamma_2, \ldots, \gamma_n \) are called linearly independent over the algebraic numbers if the dependence relation
\[
\beta_1 \gamma_1 + \beta_2 \gamma_2 + \cdots + \beta_n \gamma_n = 0
\]
with algebraic numbers \( \beta_1, \beta_2, \ldots, \beta_n \) holds only for all \( \beta_1 = \beta_2 = \cdots = \beta_n = 0 \).

Obviously, independence over the integers is equivalent to linear independence of the rationals \( \mathbb{Q} \).

**Theorem 18.1 (Lindemann-Weierstrass Theorem).** Let the algebraic numbers \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be distinct. Then the exponentials \( e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n} \) are linearly independent over the algebraic numbers. Hence
\[
\sum_{i=1}^{n} \beta_i e^{\alpha_i} \neq 0
\]
for any algebraic numbers $\beta_1, \beta_2, \ldots, \beta_n$, unless $\beta_1 = \beta_2 = \cdots = \beta_n = 0$.

The Lindemann-Weierstrass Theorem immediately yields the classical results that $e, \pi, \ln 2$ are transcendental.

- Assume towards a contradiction that Euler’s number $e$ is algebraic. We put $\alpha_1 = 0, \alpha_2 = 1, \beta_1 = -e, \beta_2 = 1$ and get the contradiction $-e \cdot e^0 + 1 \cdot e^1 \neq 0$. (Hermite 1873)

- Assume towards a contradiction that $\pi$ is algebraic. We put $\alpha_1 = 0, \alpha_2 = i\pi, \beta_1 = \beta_2 = 1$ and get the contradiction $1 \cdot e^0 + 1 \cdot e^{i\pi} \neq 0$. (Lindemann 1882)

- Assume towards a contradiction that $\ln 2$ is algebraic. We put $\alpha_1 = 0, \alpha_2 = \ln 2, \beta_1 = -2, \beta_2 = 1$ and get the contradiction $-2 \cdot e^0 + 1 \cdot e^{\ln 2} \neq 0$.

**Corollary 55.** Let the distinct algebraic numbers $\alpha_1 \neq 0, \alpha_2 \neq 0, \ldots, \alpha_n \neq 0$ be all nonzero and the algebraic numbers $\beta_1 \neq 0, \beta_2 \neq 0, \ldots, \beta_n \neq 0$ be nonzero. Then the sum

$$
\sum_{i=1}^{n} \beta_i e^{\alpha_i}
$$

is transcendental.
Theorem 18.2 (Baker 1967). Let the numbers $e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n}$ be algebraic and the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ be $\mathbb{Z}$-linearly independent. Then
\[ \sum_{i=1}^{n} \beta_i \alpha_i \] is transcendental
for any algebraic numbers $\beta_1, \beta_2 \ldots \beta_n$, unless $\beta_1 = \beta_2 = \cdots = \beta_n = 0$.

Baker’s Theorem immediately yields the classical results that $e^{\pi}$ and $2^{\sqrt{2}}$ are transcendental.

- Assume towards a contradiction that $e^{\pi}$ is algebraic. We put $\alpha_1 = \pi, \alpha_2 = i\pi, \beta_1 = -i, \beta_2 = 1$ and get the contradiction that $\beta_1 \alpha_1 + \beta_2 \alpha_2 = (-i) \cdot \pi + 1 \cdot i\pi = 0$ is transcendental.

- Assume towards a contradiction that $2^{\sqrt{2}}$ is algebraic. We put $\alpha_1 = \sqrt{2} \ln 2, \alpha_2 = \ln 2, \beta_1 = \sqrt{2}, \beta_2 = -2$ and get the contradiction that $\beta_1 \alpha_1 + \beta_2 \alpha_2 = 0$ is transcendental.

Theorem 18.3 (Gelfond-Schneider Theorem). Let $x$ be irrational and $a \neq 0, 1$ be any two algebraic numbers. Then $a^x$ is transcendental.

Proof. The numbers $\alpha_1 := \ln a$ and $\alpha_2 = x \ln a$ are by assumption $\mathbb{Z}$-linearly independent. Assume towards a contradiction that $a^x$ is algebraic. We put $\beta_1 := -x, \beta_2 := 1$ and get the contradiction that $\beta_1 \alpha_1 + \beta_2 \alpha_2 = 0$ is transcendental. $\square$

Theorem 18.4 (M. Waldschmidt). Let $x$ be irrational and $z \neq 0$ be any two complex numbers. Then at least one of the numbers $x, e^x, e^{xz}$ is transcendental.

Proof. The numbers $\alpha_1 := z$ and $\alpha_2 = xz$ are by assumption $\mathbb{Z}$-linearly independent. Assume towards a contradiction that all three numbers $x, e^x, e^{xz}$ are algebraic. We put $\beta_1 := -x, \beta_2 := 1$ and get the contradiction that $\beta_1 \alpha_1 + \beta_2 \alpha_2 = (-x) \cdot z + 1 \cdot xz = 0$ is transcendental. $\square$

Corollary 56 (Alan Baker). Under the assumption that the numbers $e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n}$ and $\beta_1, \ldots, \beta_n$ are algebraic, the sum
\[ \sum_{i=1}^{n} \beta_i \alpha_i \] is either algebraic or the sum equals zero. (The assumption that the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ are $\mathbb{Z}$-linearly independent has been dropped).

Remark. A thorough account of the results just reviewed is contained in Alan Baker’s book [3] on ”Transcendental Number Theory”. The book ”Making Transcendence Transparent” of Burger and Tubbs [9] is readable and informative, too. Finally one can recommend the internet source
http://en.wikipedia.org/wiki/Baker%27s_theorem
18.7 Which lunes are algebraic, which constructible?

We look once more at the generic figure 18.1. I call a lune algebraic for which the radius $a$ and $b$, the angle functions $\sin \alpha, \cos \alpha, \sin \beta, \cos \beta$ of the central angles, and the area of the lune are algebraic numbers.

**Theorem 18.5.** All algebraic lunes satisfy Hippocrates’ two basic assumptions

(a) The two circular sectors corresponding to the lune’s arcs have the same area.

(b) The central angles of the two circular arcs are commensurable.

**Proof.** Since the area of the kite $\square ACDB$ is algebraic, and the area of the lune $L$ is assumed to be algebraic, their difference is algebraic. Since

$$\text{area}(L) = \text{area}(S_A) + \text{area}(\square ACDB) - \text{area}(S_B)$$

we conclude that the difference of the areas of the circular sectors $S_A$ and $S_B$:

$$\text{area}(S_A) - \text{area}(S_B) = \alpha a^2 - \beta b^2$$

is algebraic, too. The angles $\alpha$ and $\beta$ at the centers $A$ and $B$, corresponding to the two arcs of the lune have to be measured in radian measure.

The assumptions imply that $e^{i\alpha}, e^{i\beta}$ and $a, b$ and the area $\alpha a^2 - \beta b^2$ are all algebraic. We use Alan Baker’s Corollary 56 with $\alpha_1 := i\alpha, \alpha_2 := i\beta$ and $\beta_1 := a^2, \beta_2 := -b^2$. We conclude that the the sum

$$\beta_1 \alpha_1 + \beta_2 \alpha_2 = \alpha a^2 - \beta b^2 = 0$$

Hence Hippocrates’ assumption (a) holds.

We now use the Theorem 18.4 of Gelfond Schneider and M. Waldschmidt with $x = \beta/\alpha$ and $z = i\alpha$. Assume towards a contradiction that $x$ is irrational. We would conclude that at least one of the numbers $\beta/\alpha, e^{i\alpha}, e^{i\beta}$ is transcendental. The latter two are algebraic by assumption and

$$\frac{\beta}{\alpha} = \frac{a^2}{b^2}$$

is algebraic, too, by the first part of the proof. Hence $\beta/\alpha$ is rational, confirming Hippocrates’ assumption (b).

**Main Theorem 29 (N.G. Tschebatorev and A.W. Dorodnov 1947).** There exist only five squarable and constructible lunes, corresponding to the cases in which $n:m$ is $2:1, 3:1, 3:2, 5:1, 5:3$.

**Remark.** The final result was obtained by A. W. Dorodnow, after many earlier partial results.
• E. Landau showed in 1903 that the $p : 1$ lune is not constructible if $p$ is a prime but not a Fermat prime.

• L. Tschakaloff showed in 1926 that the $17 : 1$ lune is not constructible, neither any $p : m$ lune where $p$ is a prime and $p > m$.

• N. Tschebotarov [37] showed in 1934 that no $n : m$ lune with $n, m$ both odd and $m > 5$ is constructible.

18.8 Irreducibility of the lune equation

The lune equation (18.4) can be written with the complex variable $z = e^{i\epsilon}$. Since

$$\frac{\sin(n\epsilon/2)}{\sin(\epsilon/2)} = z^{-(n-1)/2} \frac{z^n - 1}{z - 1}$$

we get in the new variable

$$\sqrt{m} \sin(n\epsilon/2) - (\pm \sqrt{n}) \sin(m\epsilon/2) = 0$$

$$\sqrt{m} \frac{z^n - 1}{z - 1} - (\pm \sqrt{n})z^{(n-m)/2} \frac{z^m - 1}{z - 1} = 0$$

After taking the square, we get an integer polynomial equation:

$$(18.5) \quad P_{n,m}(z) := m \left( \frac{z^n - 1}{z - 1} \right)^2 - nz^{n-m} \left( \frac{z^m - 1}{z - 1} \right)^2 = 0$$

Proposition 18.7. We assume that $p = n$ be an odd prime and $p > m$. Then the polynomial $P_{n,m}(z)$ in the squared lune polynomial equation (18.5) is irreducible over the integers.

Theorem 18.6 (L. Tschakaloff 1926). If $p$ is a prime, but not a Fermat prime, and $m < p$, the $p : m$ lune is not constructible.

Proof. We proceed similarly as in the proof of Proposition 17.19. We use the substitution $z = x + 1$. Recall that the binomial formula implies that is the new variable $x$, all coefficients of

$$R_p(x) := \frac{(1 + x)^p - 1}{x} = \sum_{k=1}^{p} \binom{p}{k} x^{k-1}$$

$$= p + \binom{p}{2} x + \binom{p}{3} x^2 + \cdots + \binom{p}{p-2} x^{p-3} + px^{p-2} + x^{p-1}$$

except the leading ones are divisible by $p$. Similarly, we substitute $z = 1 + x$ into the squared lune polynomial

$$P_{n,m}(z) = mR_p^2(x) - p(1 + x)^{p-m} R_m^2(x) = Q_{n,m}(x)$$
and use the Eisenstein criterium, given as Proposition 17.18, to show that the resulting polynomial is irreducible.

Indeed, all coefficients of \( Q_{n,m}(x) \) except the leading one are divisible by \( p \). The leading coefficient is \( m - p \) which is not divisible by the prime \( p \). The constant coefficient is \( mp^2 - mp^2 = mp(p-m) \) which is divisible by \( p \), but not by \( p^2 \). Hence all assumptions of the Eisenstein criterium are satisfied and the polynomials \( Q \) and \( P_{p,m} \) are irreducible. 

Remark. Similar results are contained in the article [30] which is translated from Postnikov’s 1963 Russian book on Galois theory. Note that we have only dealt with the most easy special case of the main theorem of N.G. Tschebatorev and A.W. Dorodnov. Biographic information is given in http://en.wikipedia.org/wiki/N._G._Chebotarev

18.9 Tschebychev polynomials

Definition 18.4 (Tschebychev polynomials). The Tschebychev polynomials of first and second kind are defined by

\[
T_n(\cos t) = \cos nt \quad \text{and} \quad U_n(\cos t) = \frac{\sin(n+1)t}{\sin t}
\]

for integers \( n = 0, 1, 2, \ldots \).

Proposition 18.8. Both polynomials \( T_n \) and \( U_n \) satisfy the same recursion formula

\[
T_{n+1} = 2xT_n - T_{n-1} \quad U_{n+1} = 2xU_n - U_{n-1}
\]

with the initial data \( T_0 = U_0 = 1 \) and \( T_1 = x \) but \( U_1 = 2x \), respectively. Both \( T_n \) and \( U_n \) are integer polynomials of degree \( n \). They are even for even \( n \), odd for odd \( n \). Moreover \( T_n(1) = 1 \) and \( U_n(1) = n+1 \) for all \( n \).

The \( T_n \) satisfy the composition formula \( T_n \circ T_m = T_{nm} \). The polynomials \( 2T_n(v/2) \) and \( U_n(v/2) \) are integer polynomials of the variable \( v \).

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
\( n \) & \( T_n \) & \( U_n \) \\
\hline
0 & 1 & 1 \\
1 & \( x \) & 2x \\
2 & \(-1 + 2x^2 \) & \(-1 + 4x^2 \) \\
3 & \(-3x + 4x^3 \) & \(-4x + 8x^3 \) \\
4 & \(1 - 8x^2 + 8x^4 \) & \(1 - 12x^2 + 16x^4 \) \\
5 & \(5x - 20x^3 + 16x^5 \) & \(6x - 32x^3 + 32x^5 \) \\
6 & \(-1 + 18x^2 - 48x^4 + 32x^6 \) & \(-1 + 24x^2 - 80x^4 + 64x^6 \) \\
7 & \(-7x + 56x^3 - 112x^5 + 64x^7 \) & \(-8x + 80x^3 - 192x^5 + 128x^7 \) \\
8 & \(1 - 32x^2 + 160x^4 - 256x^6 + 128x^8 \) & \(1 - 40x^2 + 240x^4 - 448x^6 + 256x^8 \) \\
9 & \(9x - 120x^3 + 432x^5 - 576x^7 + 256x^9 \) & \(10x - 160x^3 + 672x^5 - 1024x^7 + 512x^9 \) \\
\hline
\end{tabular}
\end{center}

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Lemma 18.6 (binomial expansion).

\[(18.6) \quad T_n(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n}{2m} x^{n-2m} (x^2 - 1)^m \]

\[(18.7) \quad U_n(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} \left( \binom{n+1}{2m+1} \right) x^{n-2m} (x^2 - 1)^m \]

Proof. Let \( x = \cos t \). Use \( e^{int} = \cos nt + i \sin nt = (\cos t + i \sin t)^n \) and separate the binomial formula into real and imaginary parts:

\[
\cos nt + i \sin nt = \sum_{k=0}^{n} \binom{n}{k} \left( \cos t \right)^{n-k} (i \sin t)^k
\]

\[
T_n(x) = \cos nt = \sum_{m=0}^{\lfloor n/2 \rfloor} \left( \binom{n}{2m} \right) (-1)^m \left( \cos t \right)^{n-2m} (\sin t)^{2m} = \sum_{m=0}^{\lfloor n/2 \rfloor} \left( \binom{n}{2m} \right) x^{n-2m} (x^2 - 1)^m
\]

\[
\sin nt = \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \left( \binom{n}{2m+1} \right) (-1)^m \left( \cos t \right)^{n-2m-1} (\sin t)^{2m+1}
\]

\[
U_{n-1}(x) = \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \left( \binom{n}{2m+1} \right) x^{n-1-2m} (x^2 - 1)^m
\]

Lemma 18.7 (generating function).

\[(18.8) \quad \sum_{n \geq 0} T_n(x) r^n = \frac{1 - xr}{1 + r^2 - 2xr} \]

\[(18.9) \quad \sum_{n \geq 0} U_n(x) r^n = \frac{1}{1 + r^2 - 2xr} \]

We see once more that both polynomials \( T_n \) and \( U_n \) satisfy the same recursion formula.

Proof. Let \( x = \cos t \). One separates a complex geometric series into real and imaginary parts:

\[
\sum_{n \geq 0} e^{int} r^n = \frac{1}{1 - e^{it} r} = \frac{1 - e^{-it} r}{|1 - e^{it} r|^2}
\]

\[
\sum_{n \geq 0} r^n \cos nt + ir^n \sin nt = \frac{1 - r \cos t + ir \sin t}{(1 - r \cos t)^2 + r^2 \sin^2 t}
\]

\[
\sum_{n \geq 0} r^n T_n(\cos t) = \frac{1 - r \cos t}{1 + r^2 - 2r \cos t}
\]

\[
r \sin t \sum_{n \geq 1} r^{n-1} U_{n-1}(\cos t) = \frac{r \sin t}{1 + r^2 - 2r \cos t}
\]
The Tschebychev polynomials of second kind are for even and odd index are even and odd, respectively.

\[ U_{2n}(x) = P_n(4x^2 - 4) \quad \text{and} \quad U_{2n+1}(x) = 2x Q_n(4x^2 - 4) \]

**Lemma 18.8.** The polynomials \( P \) and \( Q \) with the independent variable \( s = 4x^2 - 4 \) have the generating functions

\[
\sum_{k \geq 0} P_k(s) r^n = \frac{1 + s}{1 - (2 + s)r + r^2} \quad (18.10)
\]

\[
\sum_{l \geq 0} Q_l(s) r^n = \frac{1}{1 - (2 + s)r + r^2} \quad (18.11)
\]

Hence both satisfy the same the recursion formula:

\[ P_{n+2} = (2 + s)P_{n+1} - P_n \quad \text{and} \quad Q_{n+2} = (2 + s)Q_{n+1} - Q_n \]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( P_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3 + s</td>
</tr>
<tr>
<td>2</td>
<td>5 + 5s + s^2</td>
</tr>
<tr>
<td>3</td>
<td>7 + 14s + 7s^2 + s^3</td>
</tr>
<tr>
<td>4</td>
<td>9 + 30s + 27s^2 + 9s^3 + s^4</td>
</tr>
<tr>
<td>5</td>
<td>11 + 55s + 77s^2 + 44s^3 + 11s^4 + s^5</td>
</tr>
<tr>
<td>6</td>
<td>13 + 91s + 182s^2 + 156s^3 + 65s^4 + 13s^5 + s^6</td>
</tr>
<tr>
<td>7</td>
<td>15 + 140s + 378s^2 + 450s^3 + 275s^4 + 90s^5 + 15s^6 + s^7</td>
</tr>
<tr>
<td>8</td>
<td>17 + 204s + 714s^2 + 1122s^3 + 935s^4 + 442s^5 + 119s^6 + 17s^7 + s^8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n )</th>
<th>( Q_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2 + s</td>
</tr>
<tr>
<td>2</td>
<td>3 + 4s + s^2</td>
</tr>
<tr>
<td>3</td>
<td>4 + 10s + 6s^2 + s^3</td>
</tr>
<tr>
<td>4</td>
<td>5 + 21s + 21s^2 + 8s^3 + s^4</td>
</tr>
</tbody>
</table>

We put \( n = 2k \) or \( n = 2l + 1 \) into the binomial formula for \( U_{2k}(x) = P_k(s) \) or \( U_{2l+1}(x) = 2x Q_l(s) \) and get

**Lemma 18.9.**

\[
P_k(s) = 4^{-k} \sum_{m=0}^{k} \binom{2k+1}{2m+1} (4+s)^{k-m} s^m \quad (18.12)
\]

\[
2Q_l(s) = 4^{-l} \sum_{m=0}^{l} \binom{2l+2}{2m+1} (4+s)^{l-m} s^m \quad (18.13)
\]
Lemma 18.10. With the new independent variable \( s = 4x^2 - 4 \), the polynomials \( P \) and \( Q \) are monic integer polynomials with positive coefficients. The constant coefficients are \( P_k(0) = 2k + 1 \) and \( Q_l(0) = l + 1 \).

Assume now that \( p \) is an odd prime. In this case, all coefficients of the polynomial \( P_{(p-1)/2}(s) \) except the leading one are divisible by \( p \). The constant coefficient equals \( p \).

The polynomial \( P_k \) is related to the construction of a regular \( 2k+1 \)-gon. Indeed, its zeros are
\[
z_i = -4 \sin^2 \frac{i 180^\circ}{2k+1} \quad \text{for } i = 1, 2, \ldots, k
\]

Problem 18.15. Confirm that \( P_7 \) is divisible by \( P_1 \cdot P_2 \). Calculate the quotient \( P_7 / (P_1 \cdot P_2) \). Find its zeros \( z_i \) and the corresponding central angles \( i \cdot 24^\circ \) for the 15-gon. Which four integers \( i \) do you obtain, and why?

Answer. The quotient
\[
\frac{P_7}{P_1 \cdot P_2} = 1 + 8s + 14s^2 + 7s^3 + s^4
\]
have zeros \(-4 \sin^2 \frac{i 180^\circ}{15}\) for \( i = 1, 2, 4, 7 \). These are the integers less than 8 relatively prime to 15.

Proposition 18.9. Let \( p \) an odd prime. Then the polynomial \( P_{(p-1)/2}(s) \), the Tschebyshev polynomial \( U_{p-1}(x) \), and the cyclotomic polynomial \( \Phi_p(z) \) are irreducible.

Proof. We know by Lemma 18.10 that all coefficients of the polynomial \( P_{(p-1)/2}(s) \), except the leading one, are divisible by \( p \). The constant coefficient equals \( p \). By the Eisenstein criterium, given as Proposition 17.18, we conclude that the polynomial \( P_{(p-1)/2}(s) \) is irreducible.

We put the substitution \( z = e^{it} \) into the cyclotomic polynomial and get
\[
\Phi_p(z) = \frac{z^p - 1}{z - 1} = \frac{z^{(p-1)/2} - z^{(1-p)/2}}{z^{1/2} - z^{-1/2}} = \frac{\sin((p-1)t/2)}{\sin(t/2)} = U_{p-1}(\cos(t/2)) = P_{(p-1)/2}(-4 \sin^2(t/2))
\]

We see that any factoring of \( \Phi_p(z) \) into integer polynomials entails an integer factoring of the Tschebyshev polynomial \( U_{p-1}(s) \) with \( s = \cos(t/2) \), and an integer factoring of the polynomial \( P_{(p-1)/2}(s) \) with independent variable \( s = -4 \sin^2(t/2) \).

We have ruled out such a factoring. Hence the Tschebyshev polynomial \( U_{p-1}(s) \) and finally the cyclotomic polynomial \( \Phi_p(z) \) are irreducible, too.

The lune equation
\[
\frac{\sin(n\varepsilon/2)}{\sin(m\varepsilon/2)} = \pm \sqrt{n/m}
\]

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can be written with the Tschebychev polynomials of second kind

\[ U_n(\cos t) = \frac{\sin(n+1)t}{\sin t} \]

We use instead of \( x = \cos(\varepsilon/2) \) the new independent variable

\[ s = 4x^2 - 4 = -4\sin^2(\varepsilon/2) = 2\cos \varepsilon - 2 \]

The Tschebychev polynomials of second kind are expressed by the new polynomials \( P \) and \( Q \) using the independent variable \( s \).

**Lemma 18.11 (Some reduction of the lune equation).** For \( n \) and \( m \) both odd, we get the lune polynomial equation

\[
\sqrt{m} P_{(n-1)/2}(2\cos\varepsilon - 2) - (\pm\sqrt{n}) P_{(m-1)/2}(2\cos\varepsilon - 2) = 0
\]

which is of half the degree, and easier to solve. For \( n \) odd and \( m \) even, we get the original lune polynomial equation

\[
\sqrt{m} P_{(n-1)/2}(-4+4\cos^2(\varepsilon/2)) - (\pm\sqrt{n})2\cos(\varepsilon/2)Q_{(m-2)/2}(-4+4\cos^2(\varepsilon/2)) = 0
\]

The half angle needs to be used and there is no reduction of the degree.

An interesting peculiarity occurring in the case \( n = 9, m = 1 \) is mentioned by Postnikov in the survey article [30]. We get the lune polynomial equations

\[ s^4 + 9s^3 + 27s^2 + 30s + 9 + 3\sigma_0 = 0 \]

again for the variable \( s = 2\cos\varepsilon - 2 \) and \( \sigma_0 = \pm 1 \). The case with the minus sign \( \sigma_0 = -1 \) can indeed be solved by square roots! The factoring

\[ s^4 + 9s^3 + 27s^2 + 30s + 12 = ((s + 2)^2 + \omega(s + 2) - 2) \cdot ((s + 2)^2 - \omega(s + 2) - 2) \]

with \( \omega = (1 + \sqrt{-3})/2 \) leads to the four roots

\[ s_i = \frac{-9 + \sigma_1\sqrt{-3} + \sigma_2\sqrt{30 - 2\sigma_1\sqrt{-3}}}{4} \]

with \( \sigma_1 = \pm 1 \) and \( \sigma_2 = \pm 1 \). Only because these turn out to be complex roots does one not obtain another constructible lune.

The polynomial \( s^4 + 9s^3 + 27s^2 + 30s + 3 \) for the case with the plus sign \( \sigma_0 = 1 \) can be factored into quadratic polynomials by Descartes method, too. One obtains two complex solutions, and two real solutions, one of which corresponding to a lune. But the root cannot be solved by square roots, and hence the lune is not constructible.