3 Pappus', Desargues' and Pascal's Theorems

In the section on Euclidean Geometry and Ordered Fields, we explain in detail how to construct a field from the congruence classes of segments. The question whether multiplication of segment length is commutative can be translated into a question in geometry. To formulate the relevant geometrical theorem, we look once more at the figure on page 403. There are two pairs of parallel lines, resulting from the construction

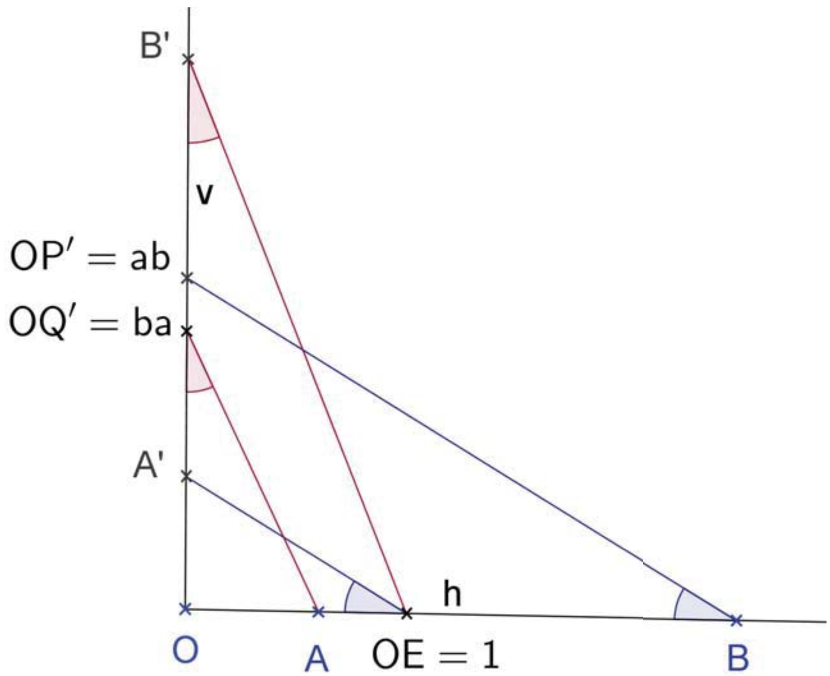

Figure 3.1: Construction of the products ab and ba, with an empirical error.

of the products ab and ba:

$$EA' \parallel BP'$$
$$EB' \parallel AQ'$$

We notice there is a third pair of parallel lines $AA' \parallel BB'$, because of the isosceles and equiangular triangles $\triangle OAA'$ and OBB'. Are the points P' are Q' equal?

Definition 3.1 (Pappus' configuration). We call the system of two intersecting lines, with three points on both of them, and the M-W hexagon zig-zaging between those triplets the *configuration of Pappus.*

In the present figure, we have points A, E and B on the horizontal line, three points

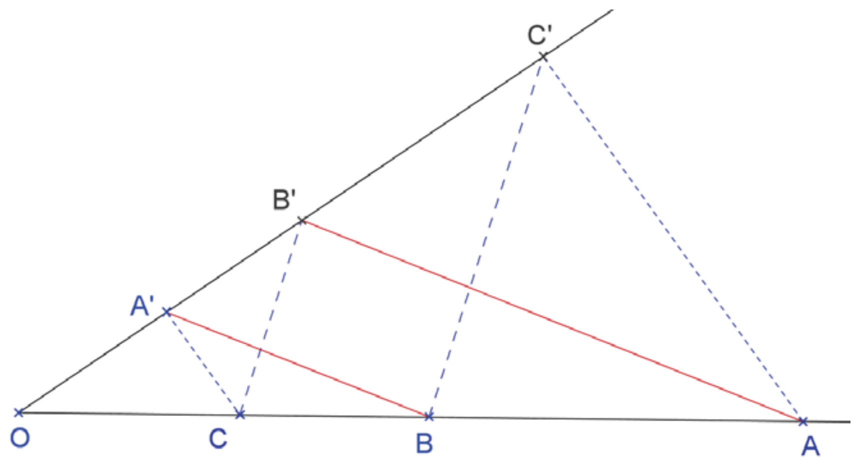

Figure 3.2: For Pappus' configuration: If $BC' \parallel B'C$ and $AC' \parallel A'C$, then $AB' \parallel A'B$.

A', P' and B' on the vertical line, and the three pairs of lines

$$\begin{array}{ccc} EA' & \text{and} & BP' \\ EB' & \text{and} & AP' \\ AA' & \text{and} & BB' \end{array}$$

the parallelism of which has to be investigated. Thus we are lead to conjecture

Theorem 3.1. *If the hexagon of Pappus' configuration contains two pairs of opposite parallel sides, the third pair of opposite sides is parallel, too.*

A formulation with definitely specified variables is given and proved as Theorem 3.3 below. For the construction of segment arithmetic, all we need is a simplified version containing two isosceles triangles.

Theorem 3.2 (Simplified Pappus' Theorem). *Let A, E, B and A', P', B' be triples of points on two intersecting lines, different from the intersection point. Assume the triangles $\triangle OAA'$ and $\triangle OBB'$ are isosceles.*

If the lines EA' and BP' are parallel, then the lines EB' and AP' are parallel, and conversely, too.

Proof. Let E' be the point on the vertical line such that $OE \cong OE'$. The four points A, B, E' and P' lie on a circle, because the angles at its opposite vertices B and E' add up to two right angles. (Explain in detail why!) Let $\mathcal{C}$ be the circle through these four points.

We have to confirm that the chords EB' and AP' are parallel.

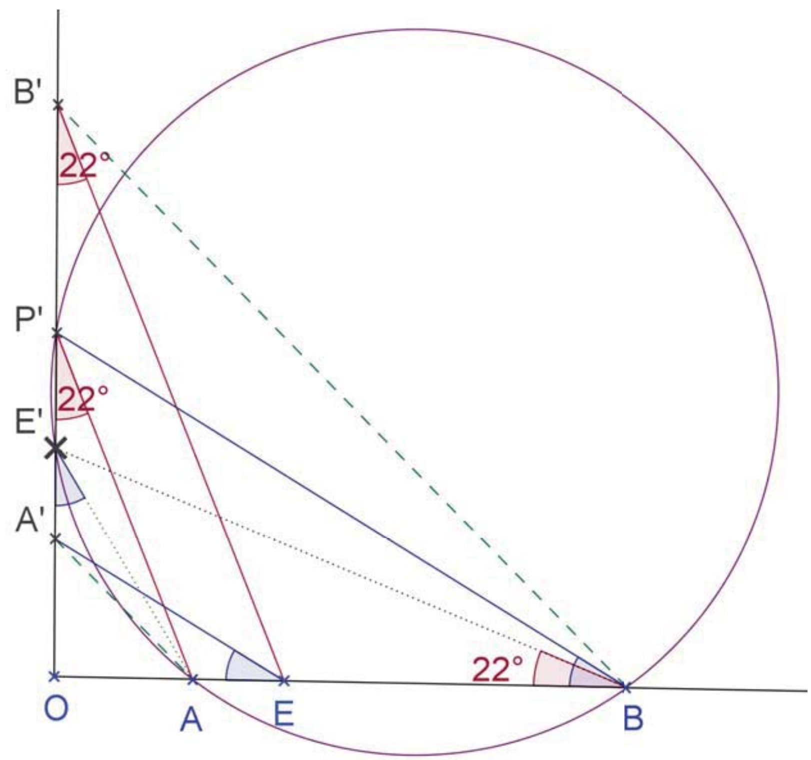

Figure 3.3: Pappus' theorem in the special case with two isosceles triangles.

For this circle, the angles $\angle E'P'A$ and $\angle E'BA$ are both circumference angles of the same arc $E'A$. By Euclid III.21 they are congruent. On the other hand, an obvious SAS-congruence implies $\angle E'BA \cong \angle EB'A'$.

By transitivity,

$$\angle OP'A = \angle E'P'A \cong \angle E'BA \cong \angle EB'A' = \angle OB'E$$

and Euclid I.27 implies the lines $P'A$ and $B'E$ are parallel. $\square$

3.1 Pappus' Theorem

Theorem 3.3 (Pappus Theorem). *Let A, B, C and A', B', C' be both three points on two intersecting lines, all different from the intersection point. If the lines BC' and $B'C$ are parallel, and the lines AC' and $A'C$ are parallel, then the lines AB' and $A'B$ and parallel, too.*

Remark. In Hilbert's foundations [22], this theorem is named after Pascal. Pascal's name is now generally associated with the theorem about the hexagon in a circle or conic section. Following Stillwell's book [35], I prefer to use the name Pappus for the theorem which does not involve any circle or conic section.

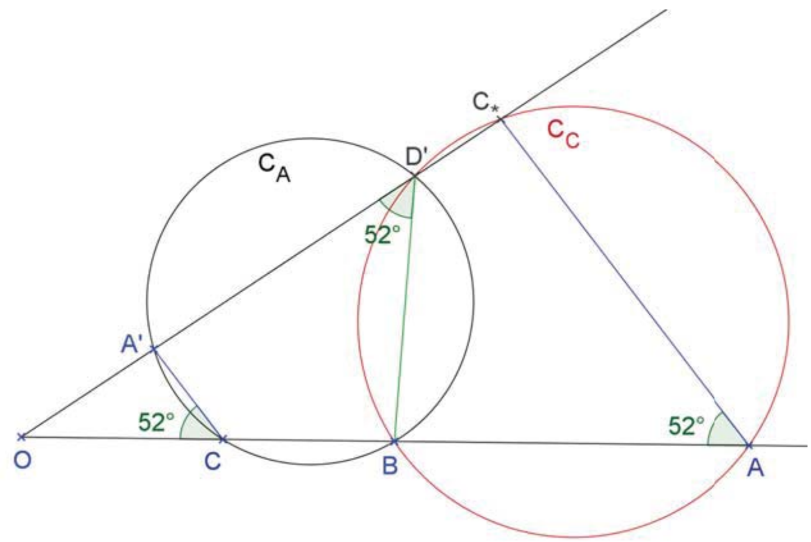

Figure 3.4: Producing parallel chords

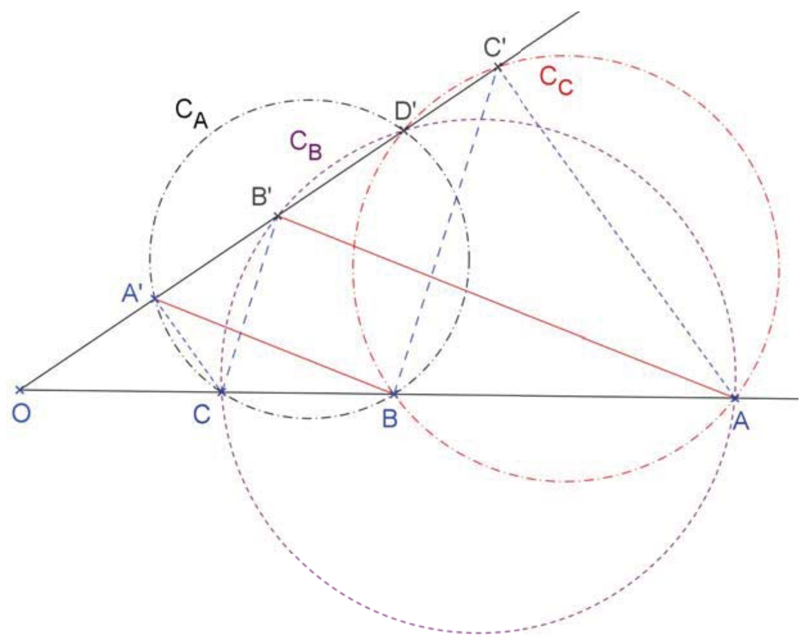

Figure 3.5: Three circles help to produce the third pair of parallel segments.

Following the third proof of Pappus' Theorem from Hilbert's foundations. Define three circles: circle $\mathcal{C}_A$ through the three points A', B and C; circle $\mathcal{C}_B$ through the three points A, B' and C; finally circle $\mathcal{C}_A$ through the three points A, B and C'. We keep the notation as in the two-circle Lemma. Let D' be the intersection point of line OA' with the circle $\mathcal{C}_A$. Let $\mathcal{C}_C^*$ be the circle through the three points D' and A, B. Finally let C^* be the intersection point of line OA' with the circle $\mathcal{C}_C^*$. Recall the two-circle Lemma 1.1:

If the endpoints of the common chord of two circles lie on two lines, these

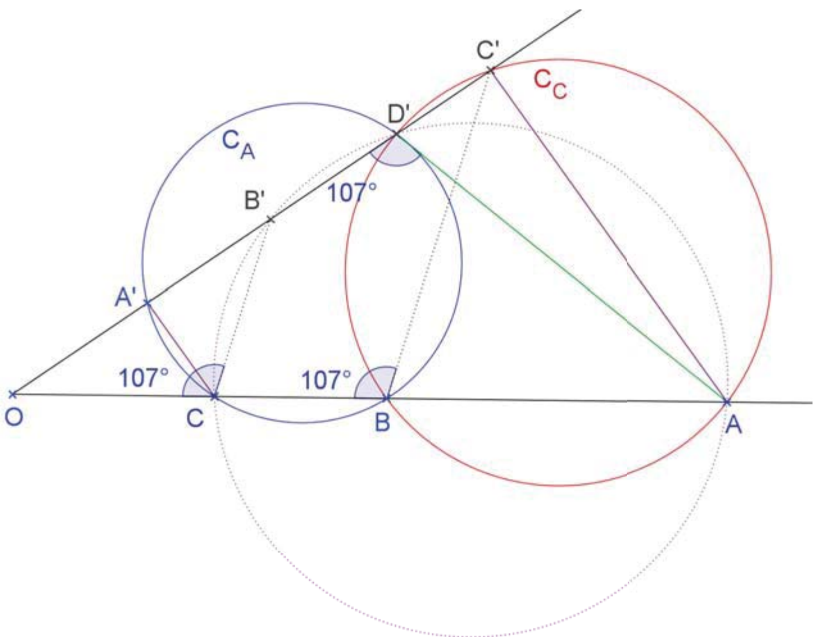

Figure 3.6: Circles $\mathcal{C}_A$ and $\mathcal{C}_C$ intersect the angle in the parallel chords $A'C$ and AC'

lines cut the two circles in two further parallel chords.

We shall use the two-circle Lemma 1.1 three times.

As a first step, the two-circle Lemma 1.1, yields that the chords $A'C$ and AC^* are parallel. Hence we conclude that $C^* = C'$, since by assumption, the chords $A'C$ and AC' are parallel, too. Hence $\mathcal{C}_C^* = \mathcal{C}_C$, and the salient point D' lies on both circles $\mathcal{C}_A$ and $\mathcal{C}_C$.

A similar reasoning is now done replacing $B \mapsto A$, $B' \mapsto A'$ and $\mathcal{C}_A \mapsto \mathcal{C}_B$. Thus one gets that the salient point D' lies on both circles $\mathcal{C}_B$ and $\mathcal{C}_C$.

Question. This fact is indeed just another instance of two-circle Lemma 1.1, and the reasoning above. Use the figure on page 390 and go over all details, once more.

Answer. Let $\mathcal{C}_B^*$ be the circle through the three points D' and A, C. Finally, let B^* be the intersection point of line OA' with this circle. By the lemma, the chords BC' and B^*C are parallel. Hence $B^* = B'$, and the four points D', B', C and A lie on a circle $\mathcal{C}_B^* = \mathcal{C}_B$.

Question. Use the Lemma 1.1 a third time, now for the circles $\mathcal{C}_A$ and $\mathcal{C}_B$, and confirm that $A'B \parallel AB'$.

Answer. Since point D' lies on all three circles $\mathcal{C}_A, \mathcal{C}_B$ and $\mathcal{C}_C$, the circles $\mathcal{C}_A$ and $\mathcal{C}_B$ have the common chord CD'. The remaining intersection points of the two lines ABC and $A'B'C'$ with these two circles yields the parallel chords $A'B \parallel AB'$.

□

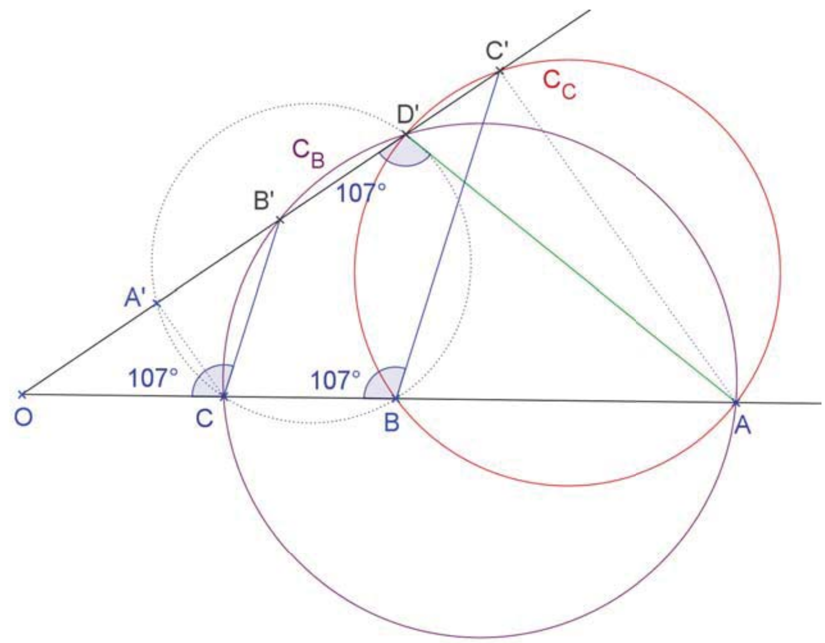

Figure 3.7: Circles $\mathcal{C}_B$ and $\mathcal{C}_C$ intersect the angle in the parallel chords $B'C$ and BC'

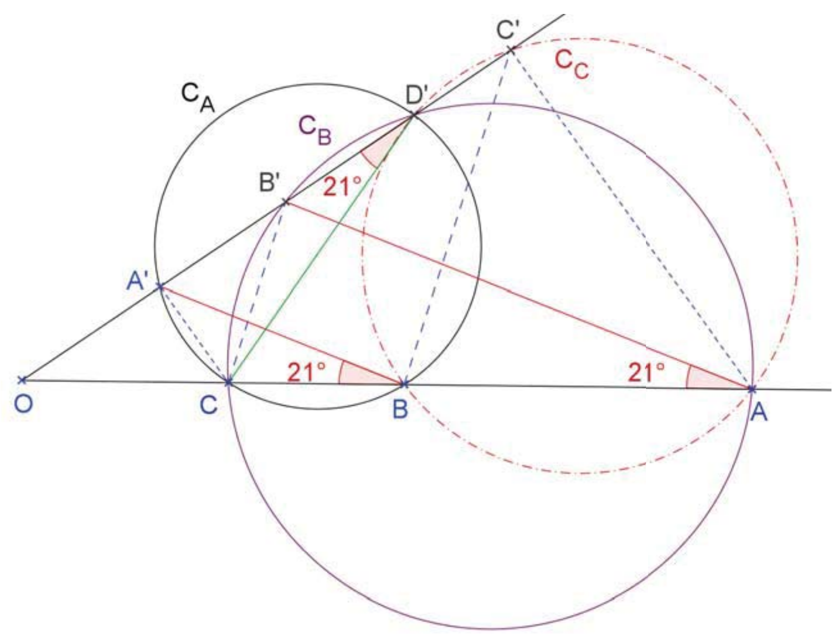

Figure 3.8: Finally, circles $\mathcal{C}_A$ and $\mathcal{C}_B$ help to confirm that AB' and $A'B$ are parallel.

3.2 Desargues' Theorem

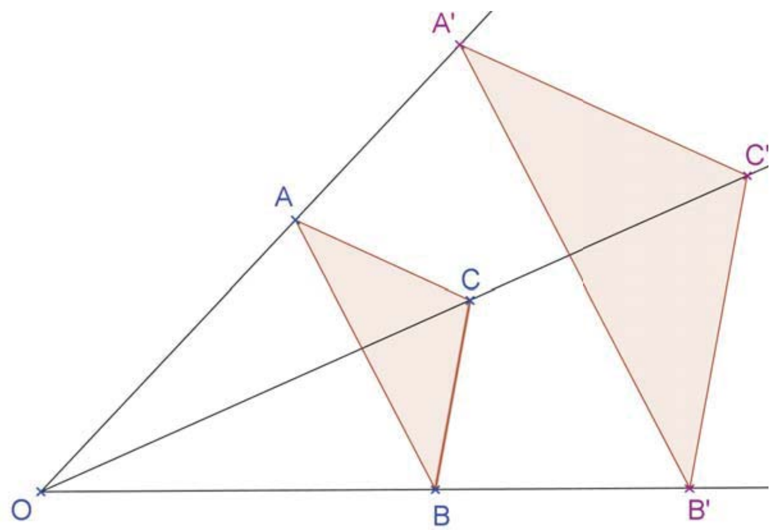

Figure 3.9: Desargues' configuration

Definition 3.2 (**Triangles in perspective**)**.** Two triangles *are in perspective from a point O* means that each of the three lines through a pair of corresponding vertices passes through point O.

Theorem 3.4 (**Desargues' Theorem**)**.** *If two triangles are in perspective, and, furthermore, two pairs of corresponding sides are parallel, then the third pair of sides are parallel, too.*

Theorem 3.5 (**Converse Desargues Theorem**)**.** *If the sides of two triangles are pairwise parallel, then the two triangles are either in perspective from a point, or the three lines through pairs of corresponding vertices are parallel.*

Question. Convince yourself that Desargues' Theorem implies the Converse Desargues Theorem.

Answer.

Question. Convince yourself that the Converse Desargues' Theorem implies the Desargues Theorem.

Answer.

There are remarkably different routes to a proof of this theorem!

Theorem 3.6 (**Theorem of Hessenberg**)**.** *For an affine plane, validity of Pappus's Theorem implies Desargues' Theorem.*

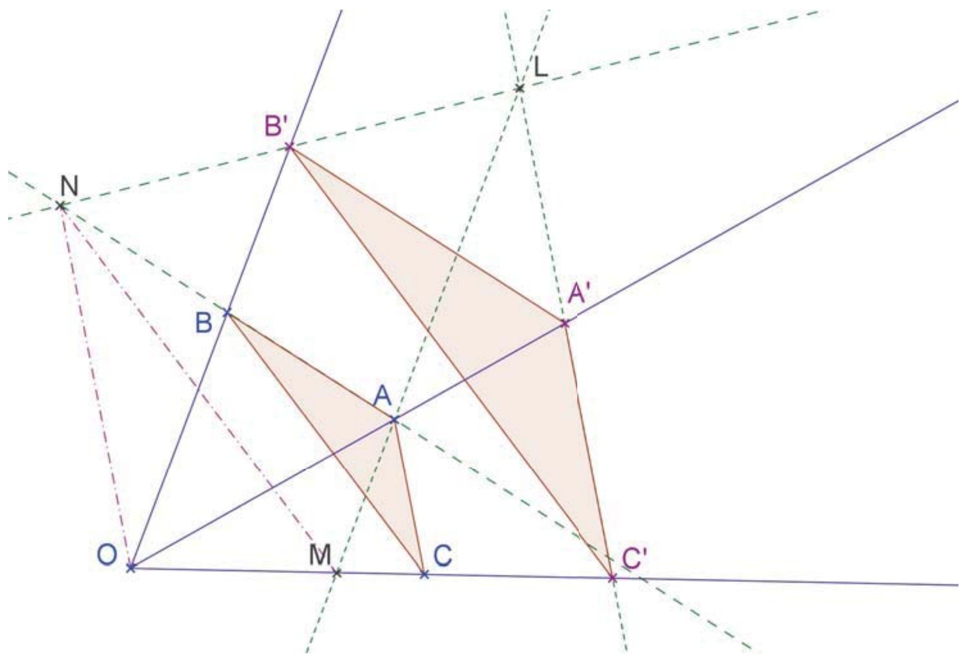

Figure 3.10: Pappus's Theorem implies Desargues' Theorem

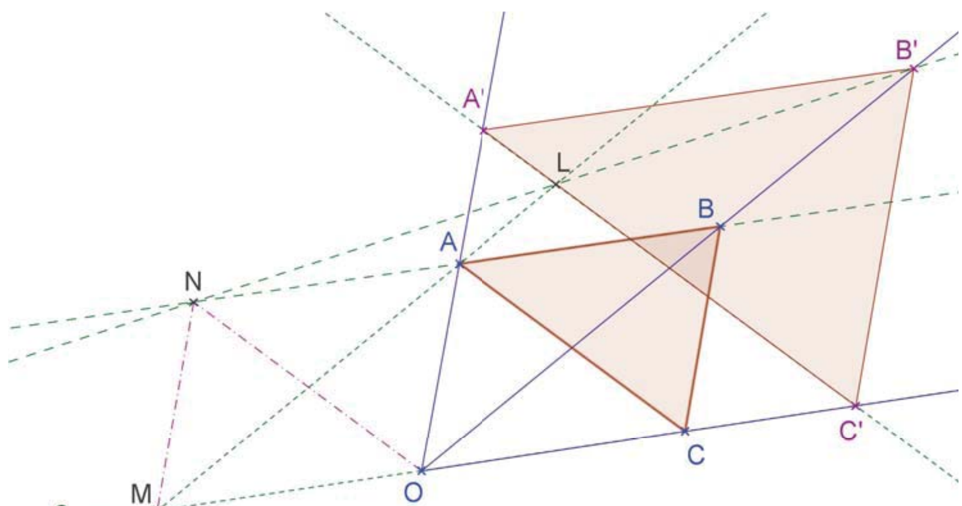

Figure 3.11: The case with $\square OABC$ a parallelogram can be handled, too

Proof. We shall proof that the second part of Desargues' Theorem holds under the given assumptions. Furthermore, we shall assume that the two triangles $\triangle ABC$ and $\triangle A'B'C'$ are in perspective from point O, and that $AB \parallel A'B'$ and $AC \parallel A'C'$. We give the proof under the following simplifying assumption:

$$OB' \nparallel A'C' \tag{3.1}$$

Draw the parallel to line OB through point A. Let L be the intersection point of this parallel with line $A'C'$. Let M be the intersection point of the parallel with line OC.

Let N be the intersection point of lines LB' and AB.

Question. Why do points L, M and N exist?

Answer.

We now use Pappus' Theorem for three different configurations:

Step 1: Use Pappus' Theorem in configuration $ONALA'B'$. Because $AB = NA \parallel A'B'$ and $AL \parallel B'O$, we conclude $ON \parallel LA'$.

Now $ON \parallel LA' = A'C'$ and $A'C' \parallel AC$ imply $ON \parallel AC$.

Step 2: Use Pappus' Theorem in configuration $ONMACB$. Because $ON \parallel AC$ and $MA \parallel BO$, we conclude $NM \parallel CB$.

Step 3: Use Pappus' Theorem in configuration $ONMLC'B'$. Because $ON \parallel LC' = LA'$ and $ML \parallel B'O$, we conclude $NM \parallel C'B'$.

Finally, $NM \parallel CB$ and $NM \parallel C'B'$ imply $CB \parallel C'B'$, as to be shown. $\square$

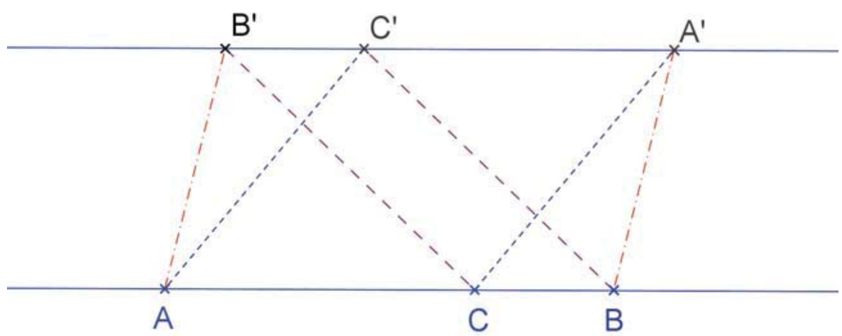

Figure 3.12: The Little Pappus Theorem asserts for a hexagon $AC'BA'CB'$ that has its vertices alternating on two parallel lines: if $BC' \parallel B'C$ and $AC' \parallel A'C$, then $AB' \parallel A'B$.

Theorem 3.7 (**Little Pappus Theorem**). *Let A, B, C and A', B', C' be both three points on two parallel lines. If the lines BC' and $B'C$ are parallel, and the lines AC' and $A'C$ are parallel, then the lines AB' and $A'B$ and parallel, too.*

Theorem 3.8 (**Little Desargues Theorem**). *If corresponding vertices of two triangles lie on three parallel lines, and, furthermore two pairs of corresponding sides are parallel, then the third pair of sides are parallel, too.*

Theorem 3.9 (**"Little Hessenberg Theorem"**). *For an affine plane, validity of the little Desargues Theorem implies the little Pappus Theorem.*

Proof. Given are points A, B, C and A', B', C' on two parallel lines such that $BC' \parallel B'C$ and $AC' \parallel A'C$.

We draw the parallel to line AC' through point B', and the parallel to line BC' through point A'. These two lines intersect in a point D. The little Desargues Theorem

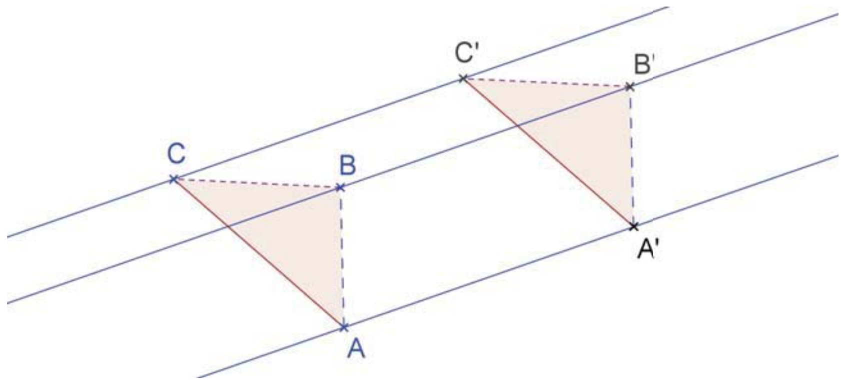

Figure 3.13: The Little Desargues Theorem asserts for two triangles with vertices on three parallel lines: if $AB \parallel A'B'$ and $BC \parallel B'C'$, then $AC \parallel A'C'$.

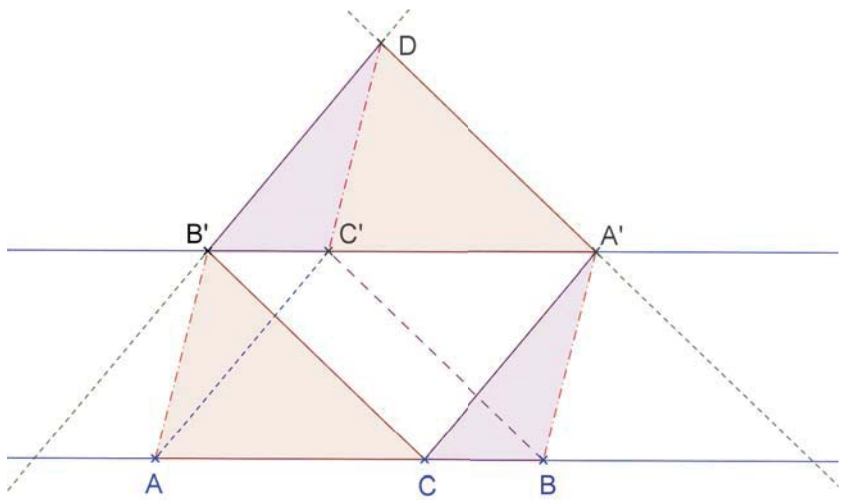

Figure 3.14: The Little Desargues Theorem implies the Little Pappus Theorem.

is now applied to the two triangles $\triangle ACB'$ and $\triangle C'A'D$. Corresponding vertices are indeed joint by three parallel lines. Too, there are two pairs of parallel sides: $AC \parallel C'A'$ and $CB' \parallel A'D$. Hence the little Desargues Theorem assures that

$$AB' \parallel C'D$$

Secondly, one applies the little Desargues Theorem to triangles $\triangle CBA'$ and $\triangle B'C'D$. Again, corresponding vertices are indeed joint by three parallel lines. Two pairs of parallel sides are $CB \parallel B'C'$ and $CA' \parallel B'D$. Hence the little Desargues Theorem assures that

$$BA' \parallel C'D$$

Both instances of the little Desargues theorem together imply $AB' \parallel BA'$, as to be shown. $\square$

Remark. For simplicity all four configurations— Pappus, Desargues and Little Pappus, Little Desargues— were given above in the *affine* version. The affine version deals with parallel lines. Hilbert's foundations [22] use the affine version.

As explained in definition 2.14, improper elements can be adjoined to produce a projective plane from a given affine plane. The bundles of parallel lines are denoted as *improper points* for the different directions of these bundles. The line through all improper points is called the *improper line*.

All four configurations and theorems— Pappus, Desargues and Little Pappus, Little Desargues— have a corresponding version in the *projective* plane. In the projective version, the statement that any lines are parallel has to be replaced by the statement that they intersect on the improper line. Stillwell's book [35] states only the projective versions.

3.3 Pascal's Theorem

Theorem 3.10 (Pascal's Hexagon Theorem). *The three pairs of opposite sides of an arbitrary circular hexagon intersect in three points lying on one line.*

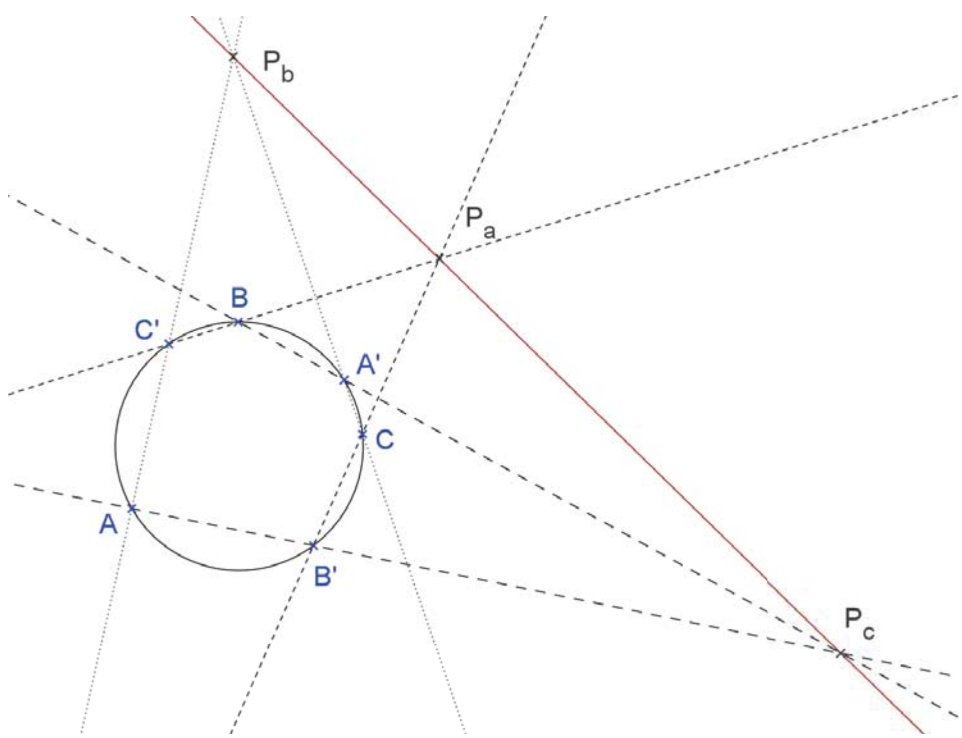

Figure 3.15: Pascal's circular hexagon

Corollary 39 (Pascal's Hexagrammum Mysticum). *The three pairs of opposite sides of an arbitrary hexagon inscribed into any conic section intersect in three points*

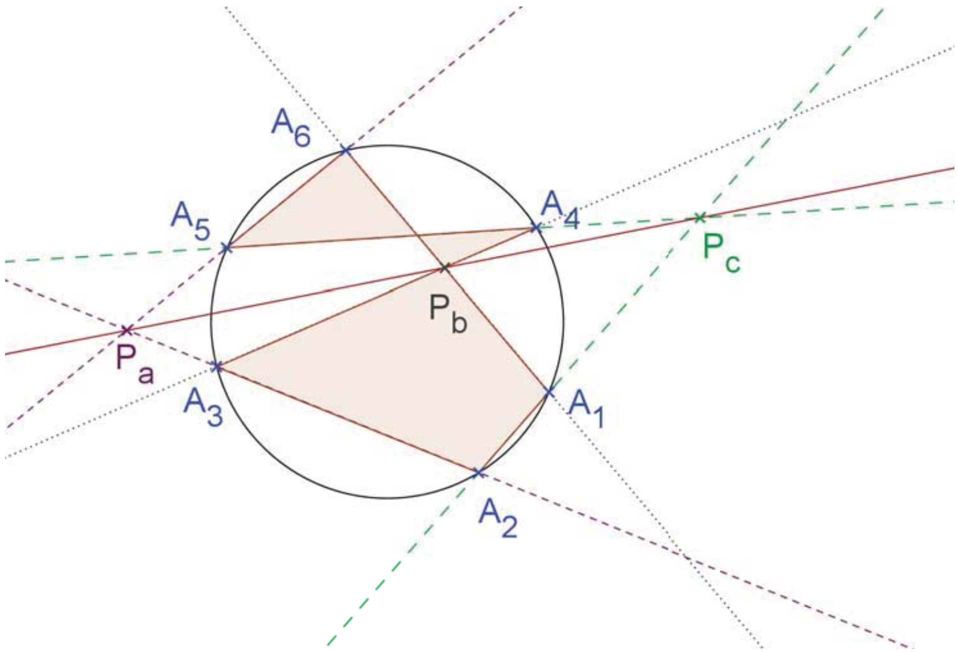

Figure 3.16: Pascal's circular hexagon tangled

lying on one line. With points and lines of the projective plane, there are no exceptional cases.

Proof. This theorem follows from Pascal's circular hexagon theorem by applying a convenient projective transformation to the configuration. Indeed, for any given conic section, there exists a projective transformation mapping the given conic section into a circle. This transformation preserves collinearity and incidence, in the projective sense. Hence the given hexagon inscribed into any conic section is mapped into a circular hexagon. By the inverse transformation, the entire configuration of Pascal's circular hexagon is mapped back into Pascal's Hexagrammum Mysticum. □

Remark. In the setting of projective geoemtry, Pascal's Hexagrammum Mysticum is really the natural common generalization of several theorems from this section. Note that both the circle, and pairs of parallel or intersecting lines are special cases for conic sections. The corresponding special cases for the Theorem of the Hexagrammum Mysticum are Pascal's circular hexagon theorem 3.10, and the projective versions of the Pappus and Little Pappus theorems, respectively. Furthermore, in the second case, choosing the Pascal line as improper line, we can further specialize to the affine versions of the Pappus theorem 3.3 and Little Pappus theorem 3.7 as stated above.

Proof of Pascal's circular hexagon. Occurring in counterclockwise order, let the circular hexagon have the vertices A, B', C, A', B, C'. Let $P_c := AB' \cap A'B$, $P_a := B'C \cap BC'$ and $P_a := CA' \cap C'A$ be the intersection points of opposite sides, extended.

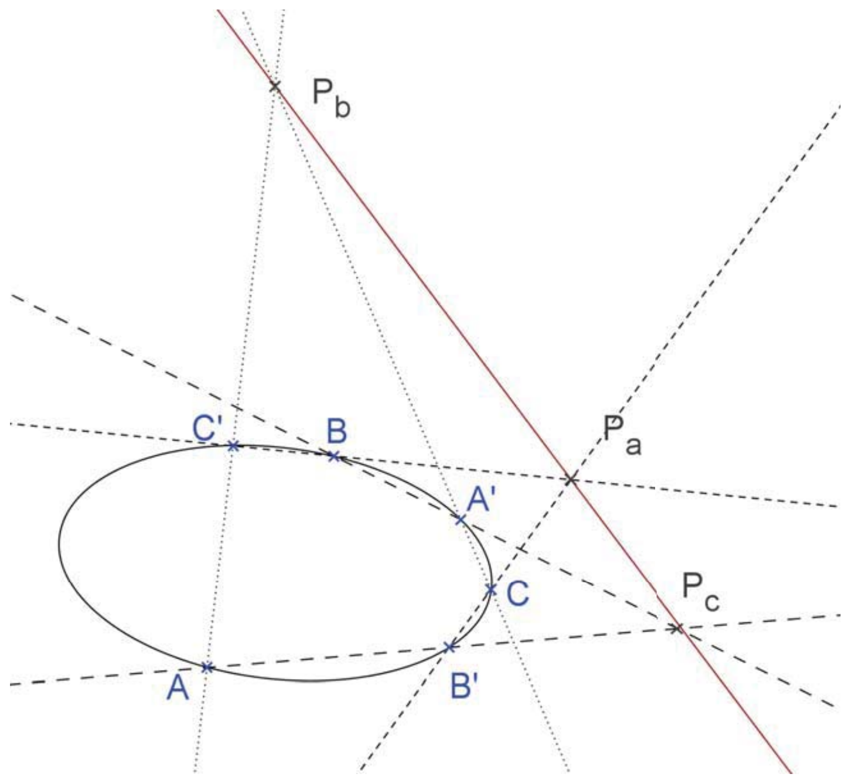

Figure 3.17: Hexagrammum Mysticum

For the proof we need a second circle $\mathcal{Q}$ through the points B, B' and P_a. Lemma 1.1 is now used with the originally given circle, and this second circle $\mathcal{Q}$. The common chord of these two circles is BB'. Through both points B and B', the given configuration has two lines drawn. Thus one is led to four instances of Lemma 1.1. One of these does not lead to a result, because lines BC' and $B'C$ intersect the circle $\mathcal{Q}$ in the same point P_a. The other three instances yield three pairs of parallel chords:

$$
\begin{aligned}
AC' &\parallel Q'P_a \\
AA' &\parallel Q'Q \\
A'C &\parallel QP_a
\end{aligned}
\tag{3.2}
$$

where Q is the second intersection point of circle $\mathcal{Q}$ with line BA', and point Q' is the second intersection of circle $\mathcal{Q}$ with line $B'A$. The segments AC' and $A'C$ are extended, and intersect in point P_b. Thus one gets two triangle $\triangle AA'P_b$ and $\triangle Q'QP_a$ for which three pairs of corresponding sides are pairwise parallel. By the converse Desargues Theorem, these two triangles are in perspective. Hence the three lines $AQ' = AB'$, $A'Q = A'B$ and P_bP_a intersect in one point, which is $P_c = AB' \cap A'B$. Hence the three point P_a, P_b and P_c lie on one line, as to be shown.

The exceptional cases because of parallel lines can be eliminated by using the projective plane. $\square$

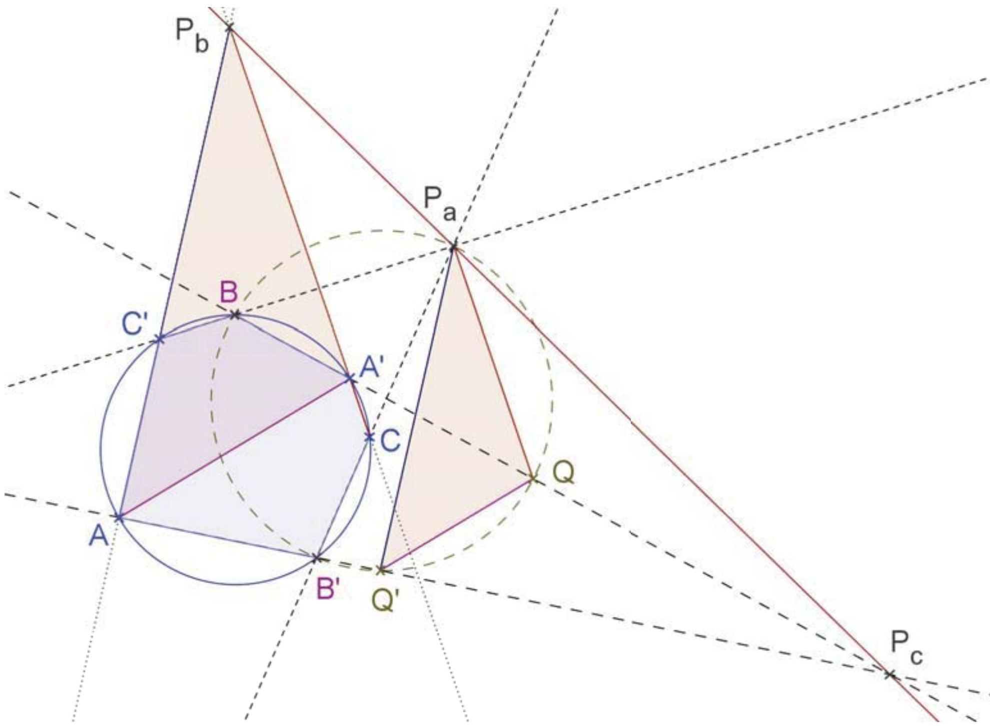

Figure 3.18: Proving Pascal's circular hexagon configuration

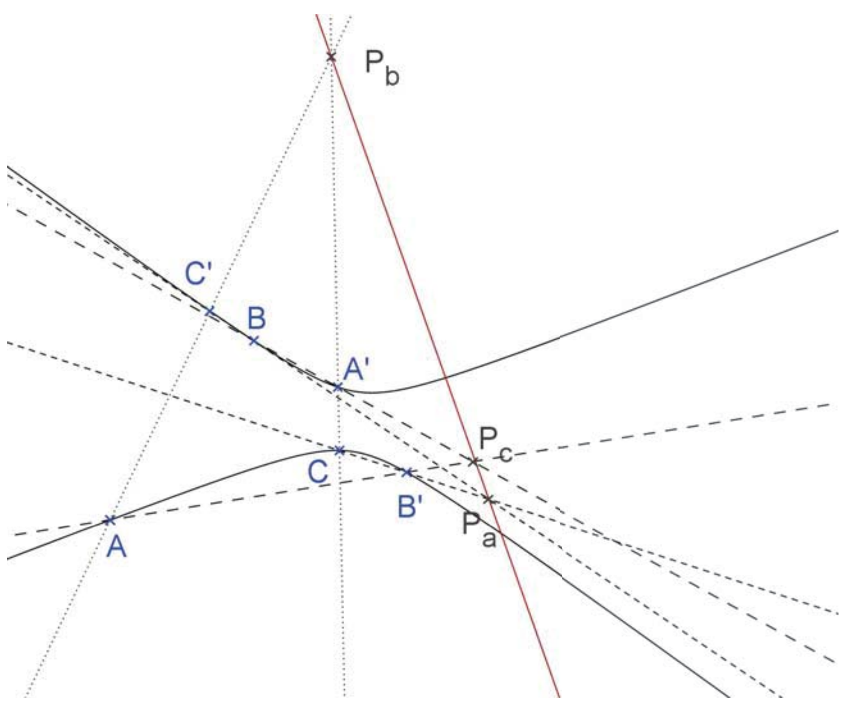

Figure 3.19: Hexagrammum Mysticum works on a hyperbola!