3 Hyperbolic Geometry in Klein's Model

3.1 Setup of Klein's model

The second important model for hyperbolic geometry goes back to Felix Klein. The reader should recall the basic idea of a model in mathematics, as explained in the passage *General remark about models in mathematics*. Again, one uses the Euclidean plane as ambient underlying reality ("background ontology"). We put into the Euclidean plane the open unit disk

$$D = \{(x, y) : x^2 + y^2 < 1\}$$

with the boundary

$$\partial D = \{(x, y) : x^2 + y^2 = 1\}$$

The center of D is denoted by O.

Definition 3.1 (Basic elements of Klein's model). The points of D are the `"points"` for Klein's model. The points of ∂D are called `"ideal points"` or `"endpoints"`. The ideal points are not points of the hyperbolic plane. Once the hyperbolic distance is introduced, the points of ∂D turn out to be infinitely far away. Hence we call ∂D the `"circle of infinity"`. The `"lines"` for Klein's model are straight chords.

Poincaré's and Klein's model differ, because lines are represented differently, and—even more importantly—the hyperbolic isometries are given by different types of mappings. In Poincaré's model, the hyperbolic reflections are realized as inversions by circles. In Klein's model, the hyperbolic reflections are realized quite differently. Indeed, hyperbolic reflections are projective mappings, which leave the circle of infinity ∂D invariant.

The developing Klein's model based on projective geometry is postponed to the subsection about the projective nature of Klein's model. I shall now use a rather simple-minded different approach: there exists an isomorphism which is a translation from Poincaré's to Klein's model. Because we already know that Poincaré's model is a consistent model for hyperbolic geometry, the translation implies that Klein's model is a consistent model for hyperbolic geometry, too.

Proposition 3.1 (The mapping from Poincaré's to Klein's model). *The point P in Poincaré's model is mapped to a point K in Klein's model by requiring that the rays $\overrightarrow{OP} = \overrightarrow{OK}$ are identical and*

$$|OK| = \frac{2\,|OP|}{1 + |OP|^2} \tag{3.1}$$

The mapping (3.1) *keeps the ideal endpoints fixed, and it takes a circular arc $l \perp \partial D$ to the corresponding chord with the same ideal endpoints. Indeed, the mapping* (3.1) *is a* translation of Poincaré's to Klein's model *, since the points and lines of Poincaré's model, are mapped to points and lines of Klein's model, preserving incidence.*

Reason. As shown in the last proposition of the section on Poincaré's model, point K is the intersection of ray $\overrightarrow{OP}$ with the chord between the ideal endpoints of any arc $l \perp \partial D$ through point P. This chord k is a hyperbolic line in Klein's model. Clearly all points of arc l are mapped to points of chord k by the same construction, and hence are all given by mapping (3.1). □

For the Poincaré disk model, it has been very useful to define *polar* elements outside the closed disk $\overline{D}$. We shall use polar elements for Klein's model, too. As a first step, we define and construct the *inverse point* P' of any given point P, in the way explained in the section about the Euclidean geometry of circles II.

Definition 3.2 (The polar elements for Klein's model). The *polar* $l^{\perp}$ of a line l is the intersection point of the tangents to ∂D at its ideal endpoints.

The *Klein polar* or *projective polar* $K^{\perp}$ of a point K is perpendicular to the ray $\overrightarrow{OK}$ at the inverse point K'.

A few clarifying remarks are in place: For both the Poincaré and the Klein model, the polar of a line is the intersection point of the tangents to the circle of infinity at the ideal ends.

But the mapping from the points to their polar elements are *different* for the two models. For clarification, I use the terms *Poincaré polar* and *projective polar*. [56] The difference occurs because the points are mapped from Poincaré's to Klein's via the isomorphism (3.1), but the polar elements are the same for both models, hence the mappings from points to their polar are different for the two models.

Proposition 3.2. *The Poincaré polar of a point P is the perpendicular bisector of the segment PP' between the given point and its inverse. The projective polar of a point K is the perpendicular to the ray $\overrightarrow{OK}$ at the inverse point K'.*

The following diagram for the mapping (3.1), *the Poincaré polar and the projective polar is commutative:*

$$\begin{array}{ccc} \textit{point } P \textit{ of Poincaré's model} & \xrightarrow{\textit{isomorphism } (3.1)} & \textit{point } K \textit{ of Klein's model} \\ {\scriptstyle \textit{perpendicular bisector of } PP'}\Big\downarrow & & \Big\downarrow{\scriptstyle \textit{perpendicular to } OK \textit{ at } K'} \\ \textit{Poincaré's polar } P^{\perp} & = & \textit{projective polar } K^{proj\perp} \end{array}$$

Proof. By item (b) from the last proposition from the section on the Poincaré model, the inverse point K' is the midpoint of the segment PP' between P and its inverse point P'. Hence the Poincaré polar $P^{\perp}$ is identical to the Klein polar $K^{\mathrm{proj}\perp}$. □

[56] The simple term polar is common usage for the projective polar. It refers to the pole and polar relation studied in projective geometry. The term is also used in common software packages for geometry. On the other hand, Goodman-Strauss uses the term polar for the Poincaré polar, as I do in the sections on Poincaré's model.

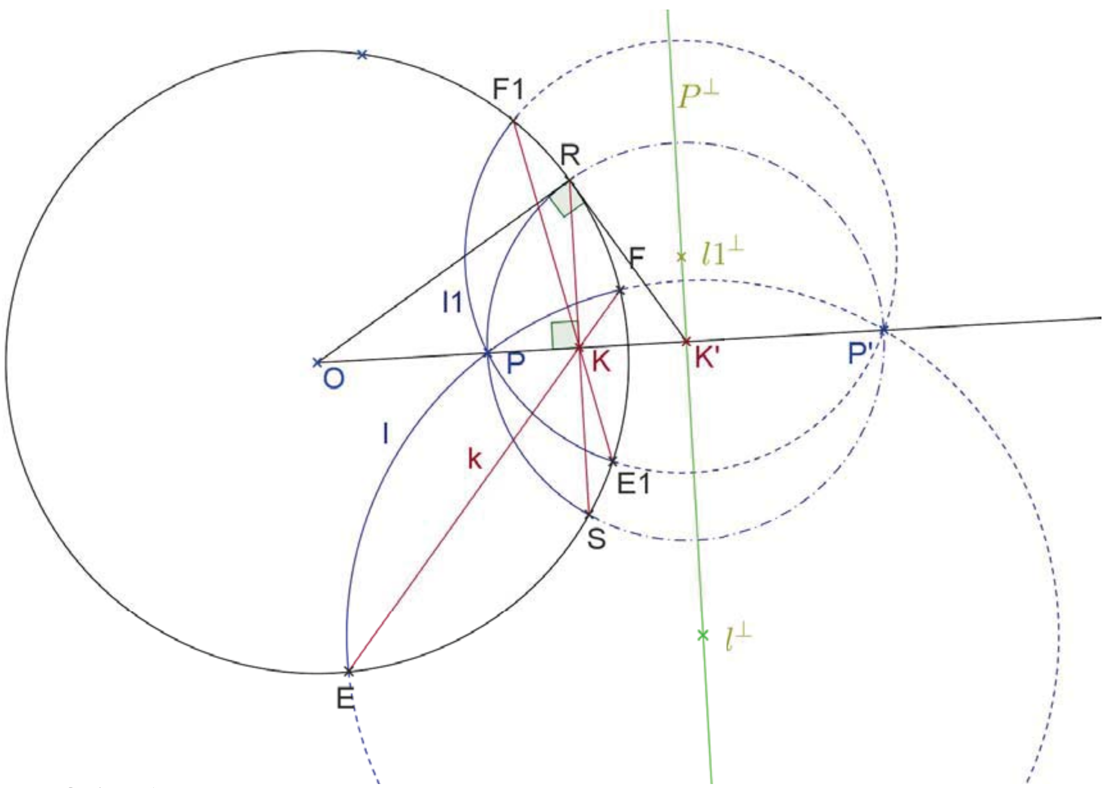

Figure 3.1: A bundle of hyperbolic lines l, l_1 through a common point P in Poincaré's model. They intersect in a common point K in Klein's model, too. Their polar elements are identical for both models.

In figure 3.1, Poincaré's elements are drawn in blue, Klein's elements in brown, and the polar elements are green. This should make clear the meaning of the commutative diagram in proposition 3.2.

Proposition 3.3. *The definitions of the projective polar of points and lines are consistent with incidence: A point K lies on a hyperbolic line k if and only if the projective polar $K^{proj\perp}$ goes through the polar $k^{\perp}$.*

Proof using the development above. As shown in the last proposition in the section about Poincaré's model, the point K lies on a chord k if and only if the Poincaré point P lies on the arc $l \perp \partial D$ with the same endpoints. This happens if and only if the polar $l^{\perp}$ lies on $P^{\perp}$. But by proposition 3.2, these polar elements are identical with those of Klein's model: $l^{\perp} = k^{\perp}$ and $P^{\perp} = K^{\text{proj}\perp}$.

Hence, expressing everything in Klein's model, we conclude that a point K lies on a hyperbolic line k if and only if the polar $K^{\text{proj}\perp}$ goes through the polar $k^{\perp}$. □

Direct independent proof. Assume that point K lies on line l. We need to check whether the polar $K^{\text{proj}\perp}$ goes through $l^{\perp}$.

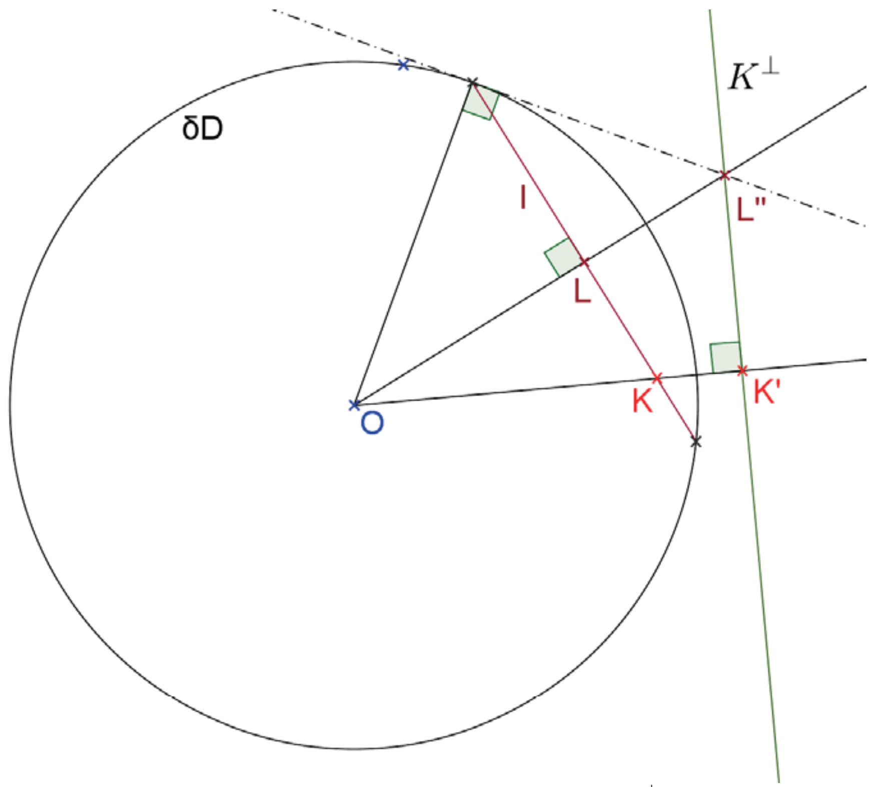

Figure 3.2: If line l goes through point K, then the polar $l^{\perp}$ lies on the polar $K^{\text{proj}\perp}$.

As drawn in figure 3.1, let L be the foot point of the perpendicular dropped from center O onto line l. The definitions of the polar and the inverse point can easily be seen to imply $l^{\perp} = L'$. Let L'' be the intersection of ray $\overrightarrow{OL}$ and the polar $K^{\text{proj}\perp}$. This construction ensures that the polar $K^{\text{proj}\perp}$ goes point L''.

The triangles $\triangle OKL$ and $\triangle OL''K'$ are equiangular. Hence, by Euclid VI.4, their sides are proportional to each other:

$$\frac{|OK|}{|OL|} = \frac{|OL''|}{|OK'|}$$

By definition of the inverse point

$$|OK| \cdot |OK'| = 1$$

and hence

$$|OL| \cdot |OL''| = 1$$

which shows that L'' is the inverse point of L. Now $L'' = L'$ and $L' = l^{\perp}$ imply $L'' = l^{\perp}$. Hence the polar $K^{\text{proj}\perp}$ goes through $l^{\perp}$, as to be shown. The converse follows as easily. □

Before discussing the metric properties and congruence, we need to clarify some terms about the use of any mathematical models, as Klein's or Poincaré's:

Definition 3.3. A theorem or a feature of a figure is part of *neutral geometry* if and only if it can be deducted assuming only the axioms of incidence, order, congruence.

The facts of neutral geometry are valid in both Euclidean and hyperbolic geometry—as well as the more exotic non-Archimedean geometries.

Definition 3.4. A feature of a figure drawn inside Klein's model (as for example an angle, midpoint, altitude or bisector) is called *absolute* if it is valid both for the underlying Euclidean plane, on which the model is based, and the hyperbolic geometry inside the model.

Remark. Here are some features that are absolute, valid both as features of hyperbolic geometry and in the underlying Euclidean plane: An angle with the center of Klein's disk appears as an absolute angle. A right angle of which one side is a diameter appears absolute. A perpendicular bisector or an angle bisector which is a diameter appears absolute.

Reason. We know that angles are depicted undistorted in Poincaré's model. For the cases mentioned above, the angles are left undistorted by the mapping from Poincaré's to Klein's model. Hence they are appear absolute in Klein's model. □

Next, we can translate orthogonality. We have shown that in Poincaré's model two hyperbolic lines l and p intersect each other perpendicularly, if and only if the polar $l^\perp$ of one line l and the ideal endpoints P and Q of the other line p lie on a Euclidean line. Since the polar of a line is easily translated, we get the following criterium for perpendicular lines in the Klein model:

Proposition 3.4 (Perpendicular lines). *In Klein's model, two hyperbolic lines l and p intersect each other perpendicularly, if and only if the polar $l^\perp$ of one line l lie on the (ultra ideal extension) of the other line p.*

Remark. Since being perpendicular is a symmetric relation, the polar $l^\perp$ of line l lies on the (ultra ideal extension) of line p if and only if the polar $p^\perp$ of the second line p lies on the (ultra ideal extension) of the first line l.

Remark. All other angles are distorted in Klein's model, and can only be defined via the isometries—which are projective mappings.

For the definition of a hyperbolic distance, one needs the cross ratio. The *cross ratio* of four point K, L, E, F is defined as

$$(KL, EF) = \frac{\overline{KE} \cdot \overline{LF}}{\overline{LE} \cdot \overline{KF}}$$

The way the endpoints E and F of the segment KL appear in this fraction can be remembered by means of the diagram:

$$\begin{array}{ll} K \to L & E \to F \\ L \leftarrow K & E \to F \end{array}$$

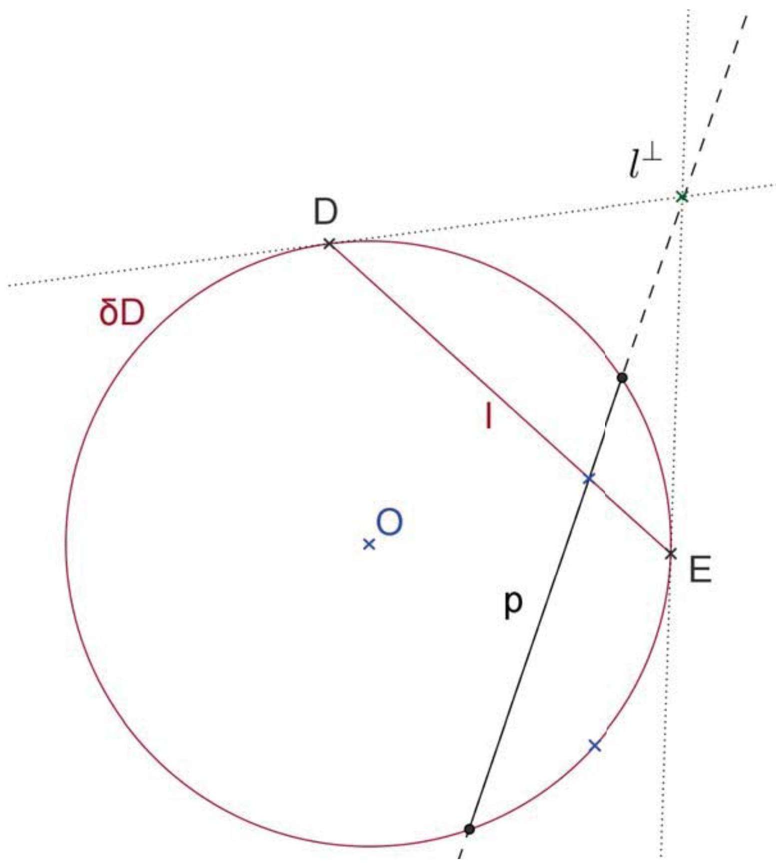

Figure 3.3: Two perpendicular lines l and p.

Definition 3.5 (Hyperbolic distance and congruence of segments). Let K, L be any two points. Let the hyperbolic line through K and L denoted by l, and the ideal endpoints of this line by E and F. We name those endpoints such that $E * L * K * F$. The hyperbolic distance or simply "`distance`" of points K and L is defined by

$$s(K,L) = \frac{1}{2}\ln(KL, EF) = \frac{1}{2}\ln\frac{\overline{KE}\cdot\overline{LF}}{\overline{LE}\cdot\overline{KF}} \tag{3.2}$$

As usual, the length of a segment is the distance of its endpoints. Two segments are called "`congruent`" if they have the same length.

3.2 Angle of parallelism

To check that formula (3.2) is the correct translation of the distance function of Poincaré's model, we use this definition of distance to derive the same formula for the angle of parallelism as is valid in Poincaré's model.

Proposition 3.5 (The angle of parallelism in Klein's model). *For any point P and line l, the angle of parallelism $\pi(s)$ relates the hyperbolic distance $s = s(P,Q)$ from*

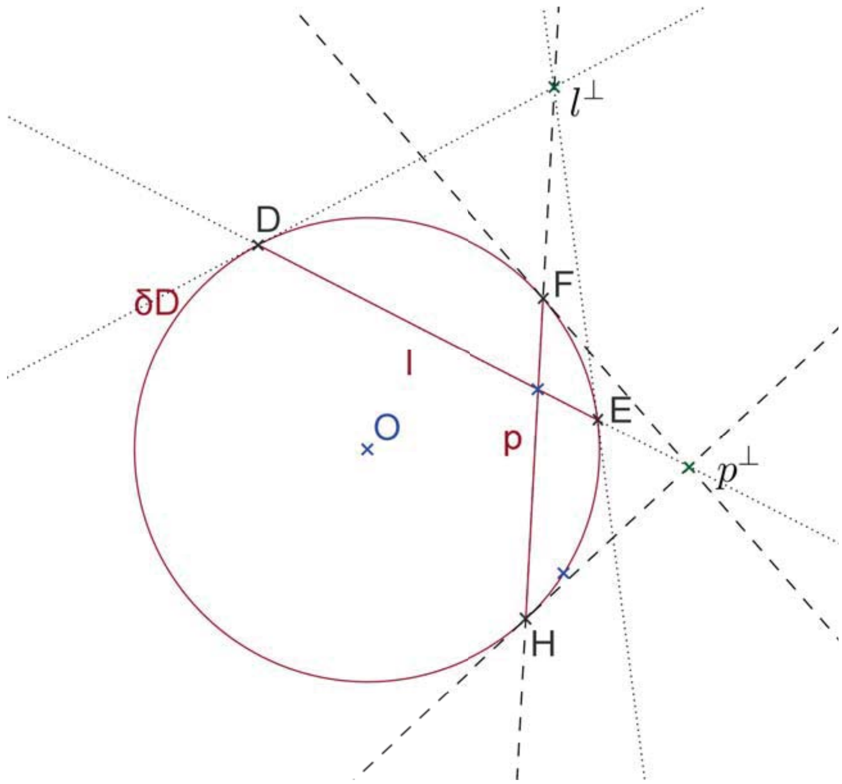

Figure 3.4: The polar of the first perpendicular line lies on the extension of the second one, and vice versa.

P to the foot point Q of the perpendicular dropped on the line l—by the remarkable Bolyai-Lobachevsky formula

$$\tan \frac{\pi(s)}{2} = e^{-s} \tag{3.3}$$

Proof. Similar to the proof for Poincaré's model, we use a special position to get easy calculations. But for Klein's model, we need to choose a different special arrangement than for Poincaré's! Because angles can be easily measured <u>only</u> at the center, we have to choose $P = O$. Moreover, we can choose the foot point Q on the positive y-axis. Thus we get at vertex Q an absolutely correct right angle between the line l and the perpendicular PQ. Let U and V be the ideal endpoints of line l, and in terms of coordinates put $V = (x, y)$. Let $Y_+ = (0, 1)$ and $Y_- = (0, -1)$ be the ideal ends of the perpendicular OQ.

From the definition (3.2) of the hyperbolic distance, we easily get

$$s = s(O, Q) = \frac{1}{2} \ln(OQ, Y_+ Y_-) = \frac{1}{2} \ln \frac{\overline{OY_+} \cdot \overline{QY_-}}{\overline{QY_+} \cdot \overline{OY_-}} = \frac{1}{2} \ln \frac{1+y}{1-y}$$

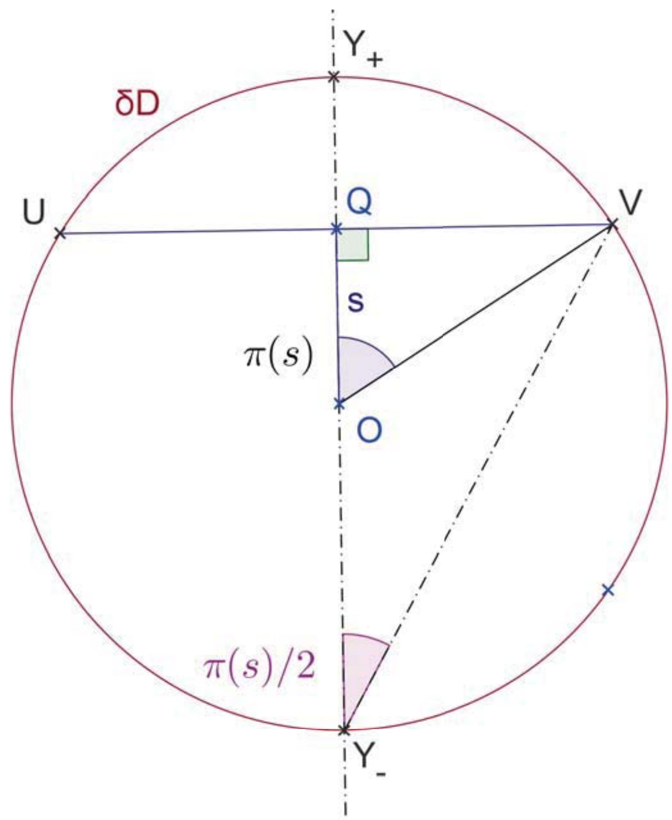

Figure 3.5: The angle of parallelism in Klein's model.

in terms of the cross ratio (OQ, Y_+Y_-) and the Euclidean distance $y = \overline{OQ}$. Applying the exponential function yields

$$e^s = \sqrt{\frac{1+y}{1-y}} \tag{3.4}$$

The angle of parallelism is defined to be the angle between the perpendicular and the asymptotic parallel. For point O and line QV, the distance point to line is $s = |OQ|$, and the corresponding angle of parallelism is $\pi(s) = \angle VOQ$. Next, we need to get $\tan \frac{\pi(s)}{2}$. By Euclid III.21, the angle at the circumference is half the angle at the center. Hence $\frac{\pi(OQ)}{2} = \angle VY_-Q$. Now the definition of the tangent function, used for the right triangle $\triangle VQY_-$, is

$$\tan \frac{\pi(s)}{2} = \frac{\overline{VQ}}{\overline{Y_-Q}} = \frac{x}{1+y}$$

Because $V = (x, y)$ is an ideal endpoint, it lies on the unit circle ∂D. Hence Pythagoras' theorem yields $x^2 + y^2 = 1$. One can eliminate x and get

$$\frac{x}{1+y} = \frac{\sqrt{1-y^2}}{1+y} = \sqrt{\frac{(1+y)(1-y)}{(1+y)^2}} = \sqrt{\frac{1-y}{1+y}}$$

and hence

$$\tan \frac{\pi(s)}{2} = \sqrt{\frac{1-y}{1+y}} \tag{3.5}$$

Because of the factor one half in the definition of the hyperbolic distance, everything fits well! Formulas (3.4) and (3.5) yield

$$\tan \frac{\pi(s)}{2} = \sqrt{\frac{1-y}{1+y}} = e^{-s}$$

which is just Bolyai's formula (3.3) for the angle of parallelism. □

As a further benefit of Klein's model, I shall confirmed Bolyai's construction of the asymptotic parallel ray.

Construction 3.1 (**Bolyai's Construction of the Asymptotic Parallel Ray).** *Given is a line l and a point P not on that line. Drop the perpendicular from P onto line l and let Q be the foot point. Erect the perpendicular onto PQ at point P. One gets a line m parallel to l. Choose a second point R on line l, and drop the perpendicular from that point onto m. Let S be the foot point. So far, we have got a Lambert quadrilateral $\square PQRS$. Now one draws a circle of radius QR around the center P. Let B be the intersection point of that circle with segment RS. Thus one gets a ray $\overrightarrow{PB}$ asymptotically parallel to the given line l.*

Reason. Similar as in proposition 3.5, we use Klein's model with disk D and put the point $P = O$ at the center of the disk. We can put the foot point Q on a vertical diameter of D.

With this layout, all three right angles of the Lambert quadrilateral $\square PQRS$ appear as right angles in Klein's model. Indeed, PQ is a vertical radius of D and PS is a horizontal radius of D, and hence the right angles at vertices P, Q and S are *absolute* right angles.

Let the line $l = QR$ have ideal ends U and V. Next, we draw the line $c = OV$. Let V' be its second ideal end. The lines RS and c intersect (why?). [57] We call the intersection point B. Thus line $c = OB$ has the ideal endpoints V and V'.

The following argument refers to the underlying Euclidean geometry (not to the hyperbolic geometry!). In the drawings, I indicate an hyperbolic right angle by a square, but a right angle for the underlying Euclidean geometry gets a doubled arc.

By Thales' theorem, $\angle V'UV$ is a right angle for the underlying Euclidean geometry. Hence, again in the underlying Euclidean geometry, the three lines UV', QP and RS

[57] Points R and S lie on different sides of line OV.

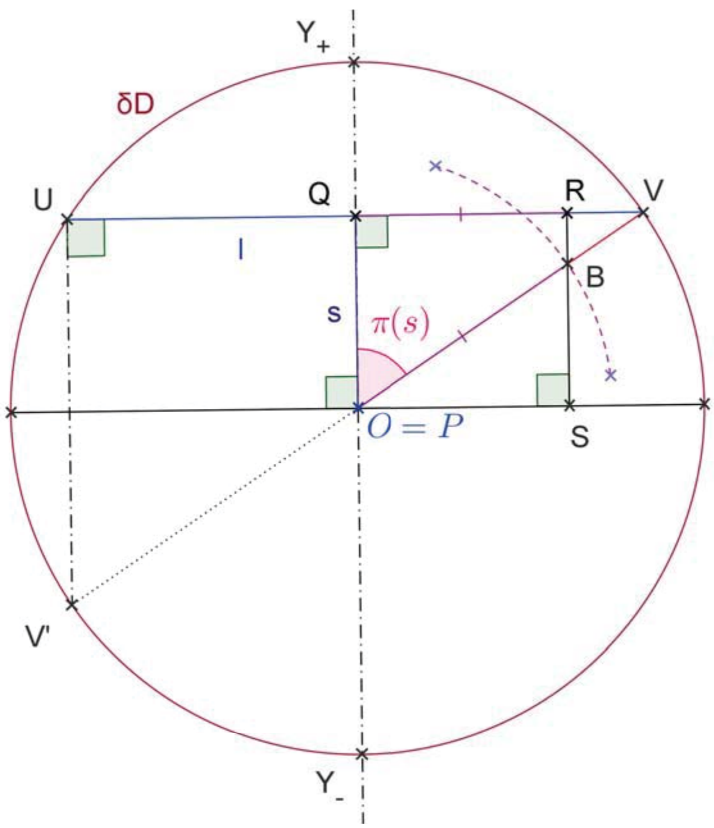

Figure 3.6: Bolyai's construction of the asymptotic parallel ray.

are parallel. Now one uses similar triangles and gets the following proportions:

$$\frac{\overline{QV}}{\overline{QU}} = \frac{\overline{OV}}{\overline{OV'}} = 1$$
$$\frac{\overline{RU}}{\overline{RV}} = \frac{\overline{BV'}}{\overline{BV}}$$
$$\frac{\overline{QV} \cdot \overline{RU}}{\overline{RV} \cdot \overline{QU}} = \frac{\overline{OV} \cdot \overline{BV'}}{\overline{BV} \cdot \overline{OV'}}$$

Hence the cross ratios and hyperbolic distances are equal, and segments QR and OB are congruent:

$$\begin{gathered}(QR, VU) = (OB, VV') \\ s(Q, R) = s(O, B) \\ QR \cong OB\end{gathered} \tag{3.6}$$

We have checked that this construction produces a line c, which has a common ideal end V with line l. Hence Bolyai's construction yields an asymptotic parallel ray $\overrightarrow{OB}$ to line l through point O. □

Because of the congruence (3.6), Bolyai's construction works *inside* the hyperbolic plane *without using the ideal endpoint* V. Indeed, the construction works *without need of a model* like Klein's or Poincaré's—and was discovered by Bolyai long before either model was known. The essential step is to draw a circle of radius QR around the center P. One get is the intersection point B of that circle with segment RS, and finally the limiting parallel OB.

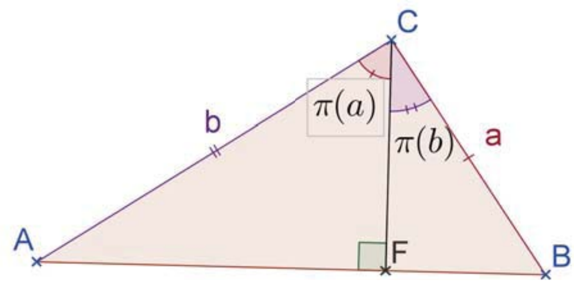

Figure 3.7: By Martin's theorem, $\angle BCF = \pi(CA)$ if and only if $\angle ACF = \pi(CB)$.

Theorem 3.1 (**George Martin's Theorem).** *If the angle between an altitude and a side of a triangle is the angle of parallelism of the other side adjacent to the vertex, then the angle between the same altitude and this second side is the angle of parallelism of the first side.*

To put it into definite terms: Let the triangle $\triangle ABC$ have acute angles at vertices A and B, and let F be the foot point of the altitude dropped from vertex C onto the side AB. In that situation,

(3.7) $$\angle BCF = \pi(CA) \quad \textit{if and only if} \quad \angle ACF = \pi(CB)$$

Proof. We can use Klein's model with center $C = O$. In that case, the angles in assertion (3.7) appear absolute, since both have vertex C. Let D be the ideal end of the perpendicular to the ray $\overrightarrow{CA}$ erected at point A, and lying on the same side as point B.

We put the Euclidean circum circle $\mathcal{C}$ around the right triangle $\triangle DAC$. We now assume that $\angle BCF = \pi(CA)$. In the drawing on page 808, this angle is named $\pi(b)$, and the corresponding segment length is $s(C, A) = b$. From the definition of the angle of parallelism, we conclude

$$\angle DCA = \pi(CA) = \angle BCF$$

Now the angle sum in the two right triangles $\triangle BCF$ and $\triangle DCA$ implies

$$\angle FBC \cong \angle ADC$$

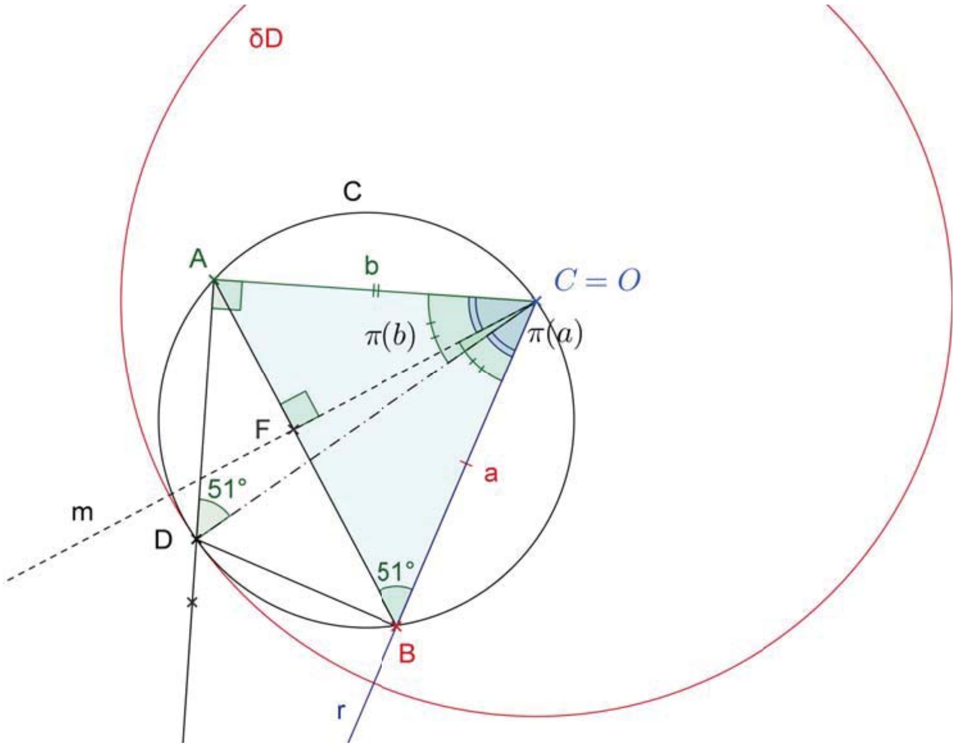

Figure 3.8: Proving Martin's theorem in Klein's model with vertex $C = O$ in the center.

Hence $\angle ABC = \angle FBC \cong \angle ADC$, and the congruence of circumference angle implies that all four points A, B, C and D lie on the circle $\mathcal{C}$— all this is true only in the Euclidean sense. Hence by Thales' theorem the angle $\angle DBC$ is right, and indeed *absolutely* right. Hence the definition of the angle of parallelism implies

$$\angle BCD = \pi(CB)$$

The sum angle $\angle BCA$ can now be partitioned in two ways, using either the interior rays $\overrightarrow{CD}$ or $\overrightarrow{CF}$. By angle additions and substraction at vertex C one arrives at

$$\angle BCA = \angle BCD + \angle DCA - \angle BCF = \pi(CB) + \beta - \beta = \pi(CB)$$

In the drawing on page 808, this angle is named $\pi(a)$, corresponding to the length $s(C, B) = a$. □

We are now able to solve the problem converse Bolyai's construction.

Problem 3.1 (Find the segment from its angle of parallelism). *Let a point C, a ray $\overrightarrow{r}$ with vertex C, and the acute angle α be given. Construct a segment CB on the given ray which has the given angle α as its angle of parallelism $\pi(CB) = \alpha$.*

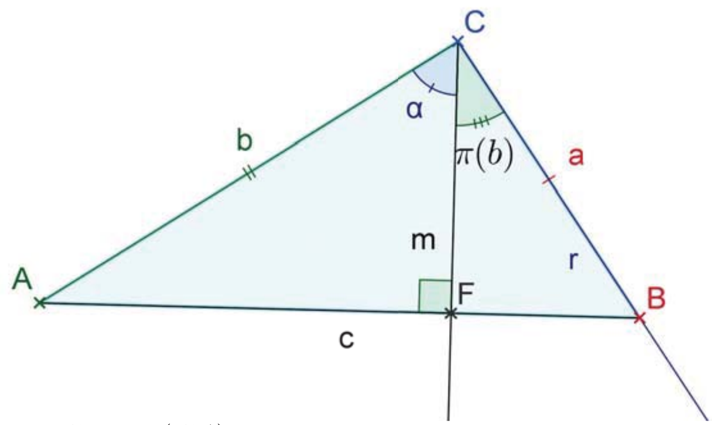

Figure 3.9: Since $\angle BCF = \pi(CA)$ by construction, Martin's theorem implies that $\angle ACF = \pi(CB)$ as requested.

Construction 3.2 (**Construction of a segment with given angle of parallelism**). *Let $CA = b$ be any other segment, for which we have already constructed the angle of parallelism $\pi(b)$, using Bolyai's construction. We transfer this angle*

$$\pi(b) = \beta$$

onto the any ray $\overrightarrow{r}$, and secondly the given angle α onto the newly produced ray, turning in the same direction. We thus obtain a third ray $\overrightarrow{CA}$ which forms with the first ray $\overrightarrow{r}$ the sum angle $\beta + \alpha$. This sum is still less than two right angles, since it is the sum of two acute angles. Onto the third ray, the segment CA of hyperbolic length $S(C, A) = b$ is transferred.

We now drop the perpendicular c from point A onto the middle ray. We obtain a foot point F on the middle ray $\overrightarrow{m}$. Indeed the perpendicular to $\overrightarrow{m}$ intersects the original ray $\overrightarrow{r}$. Let B be the intersection point. We claim that the segment CB has the angle of parallelism as required:

$$\pi(CB) = \alpha$$

The main idea. We can now use Martin's theorem: since

$$\angle BCF = \pi(CA)$$

by construction, Martin's theorem implies that

$$\angle ACF = \pi(CB)$$

as requested. □

Detailed reason for validity. The argument above does not explain, why the perpendicular c and the ray $\overrightarrow{r}$ intersect in the *hyperbolic sense*, as claimed.

We need to repeat the details for a proper justification. We can use Klein's model with center $C = O$. In that position, the angles α and β constructed above appear *absolute*, because they all have vertex C. Let $D \in \partial D$ be the ideal end of the perpendicular to the ray $\overrightarrow{CA}$ at point A, lying on the same side as foot point F. By the definition of the angle of parallelism,

$$\angle DCA = \pi(CA) = \beta$$

In the sense of Euclidean geometry, both $\angle ABC = \angle FBC = R - \pi(CA)$ and $\angle ADC = R - \pi(CA)$. The congruence of these angles $\angle ADC \cong \angle ABC$ implies that the four points A, B, C and D lie on a circle $\mathcal{C}$ in the Euclidean sense.

This circle is the Euclidean circum circle of the right triangle $\triangle DAC$. By the converse Thales theorem, this circle has diameter CD, which is a radius of Klein's disk. Hence circle $\mathcal{C}$ touches the line of infinity ∂D from inside at the ideal endpoint D. We can conclude that point B lies *inside* Klein's disk. Thus the perpendicular c and the ray $\overrightarrow{r}$ intersect in the *hyperbolic sense* in point B, as claimed.

The remaining details are easy by now: By Thales' theorem, the angle $\angle DBC$ is a right angle. This is indeed an *absolute* right angle, because its side BC goes through the center of the Klein disk. Hence, by the definition of the angle of parallelism,

$$\pi(CB) = \angle DCB = \angle ACB - \angle DCA = \alpha + \beta - \beta = \alpha$$

as to be shown. □

Problem 3.2. *For a given angle α, construct a segment with angle of parallelism $pi(a) = \alpha$. Explain Bolya's construction in elementary steps, and minimize the number of transfers needed.*

Answer. Let $\alpha = \angle ACQ$ be the given angle, with $AC \cong QC = b$ conveniently chosen. We erect the perpendiculars l at Q and p at C onto the segment QC. On the perpendicular l, we transfer a segment $QR \cong QC$, and drop the perpendicular from point R onto p. The foot-point is called S.

We draw the circle about C through points A and Q. It intersects the segment RS in point B_1. Bolya's construction tells that the rays $\overrightarrow{QR}$ and $\overrightarrow{CB_1}$ are asymptotically parallel, and hence $\angle QCB_1 = \pi(CQ)$.

We drop the perpendicular from point A onto the segment CQ. By Martin's theorem, it intersects the asymptotic ray $\overrightarrow{CB_1}$ in a point B, and the segment CB has the angle of parallelism α as requested.

Problem 3.3. *Give an illustration of this construction in Klein's model with vertex C at the center of the disk.*

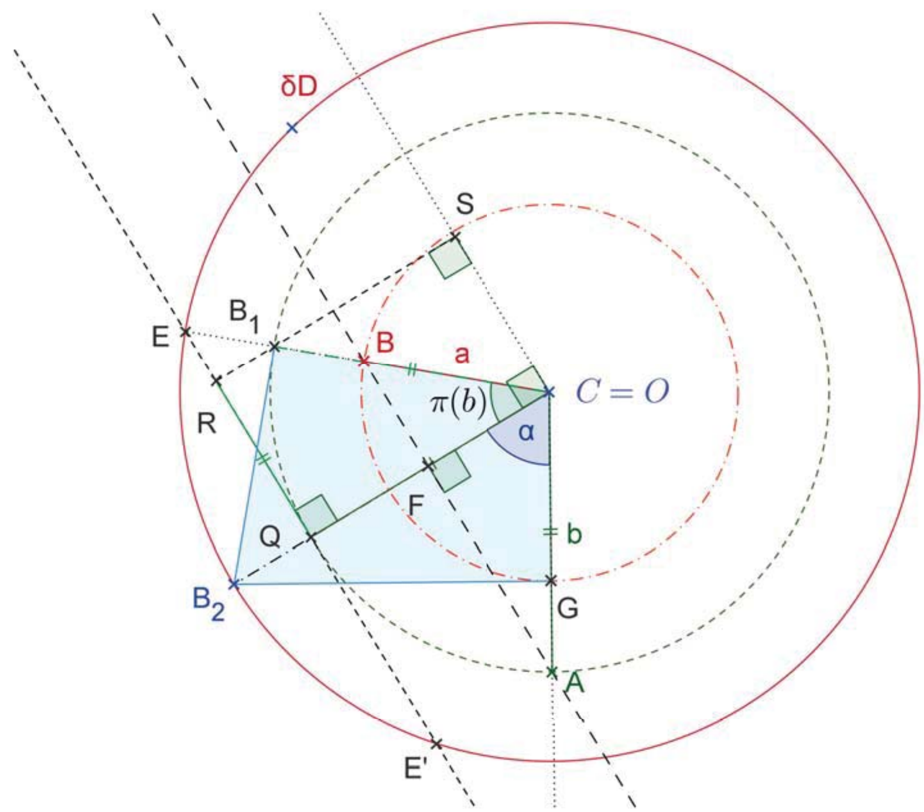

Figure 3.10: Construction of the segment $a = OB$ from given angle of parallelism α.

3.3 Projective nature of Klein's model

Definition 3.6 (The projective plane). On the set $\mathbf{R}^3 \setminus \{(0,0,0)\}$, an equivalence relation is defined by setting any two points $(x,y,z) \sim (\lambda x, \lambda y, \lambda z)$ with $\lambda \neq 0$ equivalent. The quotient space by this equivalence $\mathbf{R}^3/\sim$ is called the *projective plane* and denoted by $\mathbf{PR}^2$. Too, the points of $\mathbf{R}^3 \setminus (0,0,0)$ are called the *homogeneous coordinates* of the projective plane.

The projective plane has as an subset the *improper line* consisting of the points with the homogeneous coordinates $(x,y,0)$.

The projective plane can be viewed as the union of the usual Euclidean plane with the improper line. [58] For the points of usual Euclidean plane one can use for example as homogeneous coordinates $(x,y,1)$. For convenience, we define the z-slice plane

$$\mathcal{P}_1 := \{(x,y,1) \,:\, x,y \in \mathbf{R}\}$$

[58]Because of its improper points and line, the projective plane has a different topological structure that the Euclidean plane. Especially, the Restricted Jordan Curve Theorem 3.12 from the section on incidence geometry, is not valid for the projective plane.

For any nonsingular 3×3 matrix A, the linear mapping $x \in \mathbf{R}^3 \mapsto Ax \in \mathbf{R}^3$, induces a mapping $\phi_A : x \in \mathbf{PR}^2 \mapsto x' = Ax \in \mathbf{PR}^2$, by means of the homogeneous coordinates.

For the points of the Euclidean plane, this mapping is given by the fractional linear transformations

$$x_1' = \frac{a_{11}x_1 + a_{12}x_2 + a_{13}}{a_{31}x_1 + a_{32}x_2 + a_{33}} \quad \text{and} \quad x_2' = \frac{a_{21}x_1 + a_{22}x_2 + a_{23}}{a_{31}x_1 + a_{32}x_2 + a_{33}} \tag{3.8}$$

The mapping of the points on the improper line can be rather easily deduced from these formulas. The details can be left to the reader.

Definition 3.7 (Projective mapping). The extensions of the fractional linear mappings (3.8) with $\det A \neq 0$, to the projective plane are called *projective mappings*.

Main Theorem 32. *Given are four points $x_1, x_2, x_3, x_4 \in \mathbf{PR}^2$ with no three of them lying on a line. Similarly, there are given any four image points $x_1', x_2', x_3', x_4' \in \mathbf{PR}^2$. Once more, it is assumed that no three of them lie on a line. Then there exists exactly one projective mapping which takes x_i to x_i' for $i = 1, 2, 3, 4$.*

Proof. Let $x_1, x_2, x_3, x_4 \in \mathbf{R}^3 \setminus \{0\}$ be any homogenous coordinates of the four given points $x_i \in \mathbf{PR}^2$. Since any four vectors in $\mathbf{R}^3$ are linearly dependent, there exists $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbf{R}$ such that

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0$$

Since by assumption no three of the four points x_i lie on a line, *all four* $\lambda_i \neq 0$ are nonzero. The matrix $A = [\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3]$ maps

$$p_1 = (1,0,0) \mapsto \lambda_1 x_1 \ , \ p_2 = (0,1,0) \mapsto \lambda_2 x_2 \ , \ p_3 = (0,0,1) \mapsto \lambda_3 x_3 \ ,$$
$$p_4 = (1,1,1) \mapsto -\lambda_4 x_4 \ ,$$

and is nonsingular. The induced projective mapping $\phi_A : x \in \mathbf{PR}^2 \mapsto x' = Ax \in \mathbf{PR}^2$, takes the points $f_i \mapsto x_i$ for $i = 1, 2, 3, 4$.

By means of composition $A' \circ A^{-1}$, we find a projective mapping such that

$$x_i \in \mathbf{PR}^2 \mapsto x_i' \in \mathbf{PR}^2$$

where the four preimages x_i and images x_i' can be arbitrarily prescribed. □

Main Theorem 33 (Projective invariance of the cross ratio). *Let P_1, P_2, P_3, P_4 be any four points on a line and Q_1, Q_2, Q_3, Q_4 their images by a projective mapping. Then the four points Q_i lie on a line, too. The cross ratios $(P_1P_2, P_3P_4) = (Q_1Q_2, Q_3Q_4)$ are equal.*

Definition 3.8. The projective mappings which leave the circle of infinity ∂D invariant are called *automorphic collineations*.

With composition of mappings as group operation, the automorphic collineations form a group. The following fact shows that it is a rather large group.

Proposition 3.6. *Given is any point A inside the disk D and its image A' inside the disk. Too, there is given any ideal point U on the circle of infinity and its image U', again on the circle of infinity. Then there exist exactly two automorphic collineations mapping A to A' and U to U'. One of these two mappings preserves the orientation, the other one reverses the orientation of the hyperbolic plane.*

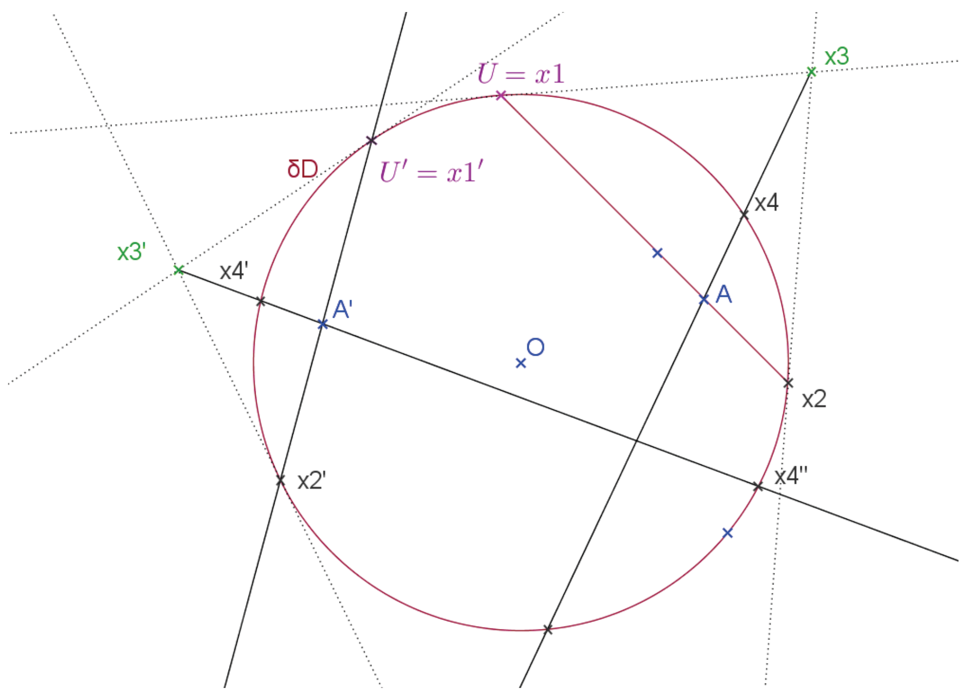

Figure 3.11: There exist two automorphic collineations mapping A to A' and U to U'.

Proof. Let $x_1 := U$ and x_2 be the second ideal end of the chord UA. Let x_3 be the intersection point of the tangent to the circle of infinity ∂D at points x_1 and x_2. Let x_4 be one of the intersection points of the line of ∂D with the line Ax_3. No two of the four points x_1, x_2, x_3, x_4 lie on a line.

The same construction has to be done with primed image points. By the Main Theorem 32, there exists a projective mapping taking x_i to x_i' for $i = 1, 2, 3, 4$. This mapping takes the tangents to ∂D at points x_i to the tangents at x_i' for $i = 1$ and $i = 2$. Counting multiplicity, five points of the circle of infinity are mapped to five other points of this line. Since a projective mapping takes a conic section to a conic section, and a conic section is determined by five of its points, [59] the entire circle of infinity is mapped to itself. Hence the projective mapping constructed above is an automorphic collineation.

[59]This fact can be shown using Pascal's magic hexagon.

Because of the arbitrary choice of x_4' among the two intersections of line $A'x_3'$ and ∂D, one gets two solutions. These are the only solutions, because lines are mapped to lines and tangents to ∂D are mapped to tangents to ∂D. □

Proposition 3.7. *Given any three different ideal points U, V, W on the circle of infinity and three different images U', V', W', again on the circle of infinity. Then there exists* exactly one *automorphic collineation mapping $U \mapsto U', V \mapsto V', W \mapsto W'$. It depends on the order of the given points, whether the mapping preserves or reverses the orientation of the hyperbolic plane.*

Answer (Proof of Proposition 3.7). Let x_1 be the intersection point of the tangents at U and V to ∂D, and x_1' be the intersection point of the tangents at U' and V' to ∂D. Of the four points x_1, U, V, W no three lie on a line, and similarly for the points x_1', U', V', W'

By the Main Theorem 32, there exists exactly one projective mapping taking $x_1 \mapsto x_1'$ and $U \mapsto U', V \mapsto V', W \mapsto W'$. This mapping takes the tangents to ∂D at points U and V to the tangents at U' and V'. Counting multiplicity, five points of the circle of infinity are mapped to five other points of this line. Since a projective mapping takes a conic section to a conic section, and a conic section is determined by five of its points, the entire circle of infinity is mapped to itself. Hence the projective mapping constructed above is an automorphic collineation.

On the other hand, the obtained mapping is unique, since any automorphic collineation with the required property necessarily maps x_1 to x_1'.

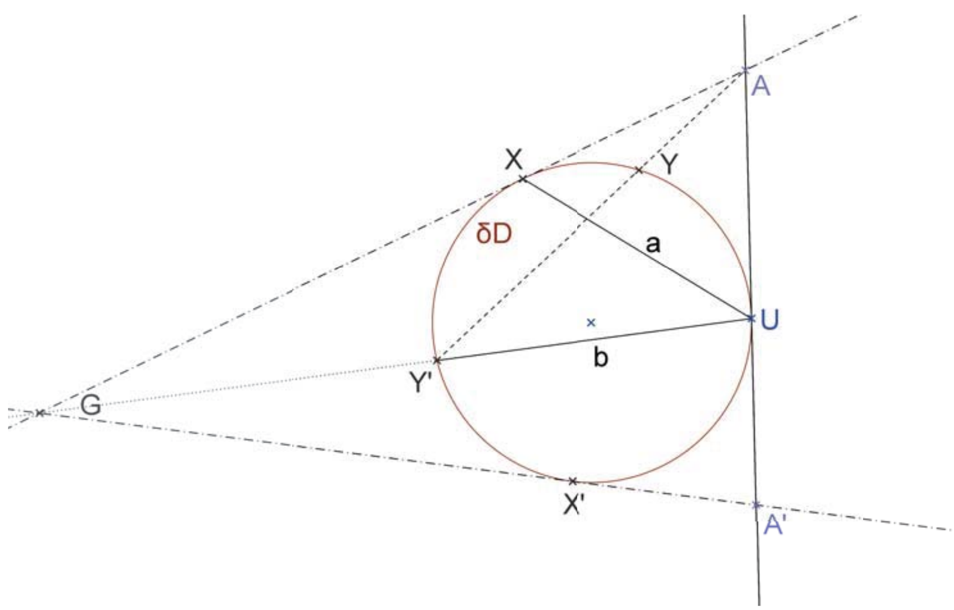

Figure 3.12: An automorphic collineation that transports points along horocycles.

Proposition 3.8. *Given any ideal point U on the circle of infinity and any two different points A and A' on the tangent to the circle ∂D at point U. Then there exists exactly one automorphic collineation which keeps U fixed, map A to A' and has no other fixed point. This mapping preserves the orientation of the hyperbolic plane.*

Proof of Proposition 3.8. Let X and X' be the touching points of the tangents from points A and A' to ∂D. We let Z' be the intersection point of these tangents and draw the line $b = UZ'$. Let Y' be the second end of line b, let Y be the second end of line AY'.

Of the four points A, X, Y, U nor of the four points A', X', Y', U no three lie on a line. By the Main Theorem 32, there exists exactly one projective mapping taking

$$A \mapsto A',\ X \mapsto X',\ Y \mapsto Y', \quad \text{and leaving point } \ U \mapsto U$$

fixed. This mapping takes the tangents to ∂D at points U and X to the tangents at U and X' since A is mapped to A'. Furthermore point $Y \in \partial D$ is mapped to $Y' \in \partial D$. Counting multiplicity, five points of the circle of infinity are mapped to five other points of this circle. Hence the prescribed mapping is an automorphic collineation.

On the other hand, the obtained mapping is unique, since any automorphic collineation with the required property necessarily maps $A \mapsto A', X \mapsto X', Y \mapsto Y'$ and leaving point $U \mapsto U$ fixed. It is left to the reader to check that X, Y, U and X', Y', U define the same orientation of the circle ∂D. □

Problem 3.4. *Convince yourself that the automorphic collineation constructed above is the composition $R_b \circ R_a$ of the two reflections across lines a and b.*

Solution. The reflection R_a takes $X \mapsto X, Y \mapsto Y'$, leaving point $U \mapsto U$ fixed. The reflection R_b takes $X \mapsto X'$, $Y' \mapsto Y'$, leaving point $U \mapsto U$ fixed. Hence the composition $R_b \circ R_a$ takes

$$X \mapsto X',\ Y \mapsto Y', \quad \text{leaving point } \ U \mapsto U$$

fixed. Since it is an automorphic collineation and preserves the orientation, it is uniquely specified by this property. Since the given mapping S is an orientation preserving automorphic collineation with the same property, we conclude $S = R_b \circ R_a$. □

The development of Klein's model is based on the fact:

Main Theorem 34. *The automorphic collineations are the isometries of the hyperbolic plane. They leave the hyperbolic distances and angles invariant.*

Proof. The invariance of the distances is an easy consequence of the projective invariance of the cross ratio. The hyperbolic distance any two points K and L is defined by

$$s(K, L) = \frac{1}{2} \ln(KL, EF) = \frac{1}{2} \ln \frac{\overline{KE} \cdot \overline{LF}}{\overline{LE} \cdot \overline{KF}} \tag{3.9}$$

where E and F are the ideal endpoints of the line KL, ordered such that $E * L * K * F$.

Any automorphic collineation maps these four points to points E', L', K', F', lying again on one line, and E' and F' lying on the circle of infinity ∂D. Hence the latter two points are the ideal ends of the image line $K'L'$ and have the hyperbolic distance

$$s(K', L') = \frac{1}{2} \ln(K'L', E'F') = \frac{1}{2} \ln \frac{\overline{K'E'} \cdot \overline{L'F'}}{\overline{L'E'} \cdot \overline{K'F'}} \tag{3.10}$$

Hence the invariance of the cross ratio by projective mappings implies

$$\frac{\overline{KE} \cdot \overline{LF}}{\overline{LE} \cdot \overline{KF}} = \frac{\overline{K'E'} \cdot \overline{L'F'}}{\overline{L'E'} \cdot \overline{K'F'}}$$

and hence $s(K, L) = s(K', L')$, as to be shown. □

The invariance of the angles allows for measurement of angles. To this end, one maps the given angle by an automorphic collineation, taking its vertex A to the center O. Because the image angle has its vertex at the center, it appears as an absolute angle and can be measured by the geometry of the Euclidean plane.

Secondly the construction used in the proof of proposition 3.6 confirms once more the criterium for right angles given in proposition 3.4.

Our next goal is the construction of a hyperbolic reflection. It turns out that the key figure is the ideal quadrilateral.

Proposition 3.9 (An ideal quadrilateral produces five right angles). *Let the ideal quadrilateral $\square ABCD$ have diagonals intersecting at P, and drop perpendicular l from point P onto side AB. This perpendicular l is perpendicular to both opposite sides AB and CD. Next we erect the perpendicular p onto l at point P. This second perpendicular p is perpendicular to the other two opposite sides BC and DA.*

Furthermore, the three lines AB, CD and p meet in one ultra ideal point, and the three lines BC, DA and l meet in another one.

Proof. There exists an automorphic collineation that maps the point P to the center $P' = O$.

Hence the special position with the intersection of the diagonals at the center of the disk can be achieved by means of an automorphic collineation, which is an isometry of the hyperbolic plane.

But in that special position, the diagonals and the lines l and p all intersect at the center O. Thales' theorem now implies that the ideal quadrilateral $\square ABCD$ appears as a rectangle. The five right angles appear as *absolute* right angles, valid both in the underlying Euclidean plane and in hyperbolic geometry. □

Reason via the Poincaré model. Put the figure into the Poincaré model, and use the special position with the intersection of the diagonals at the center of the disk. As

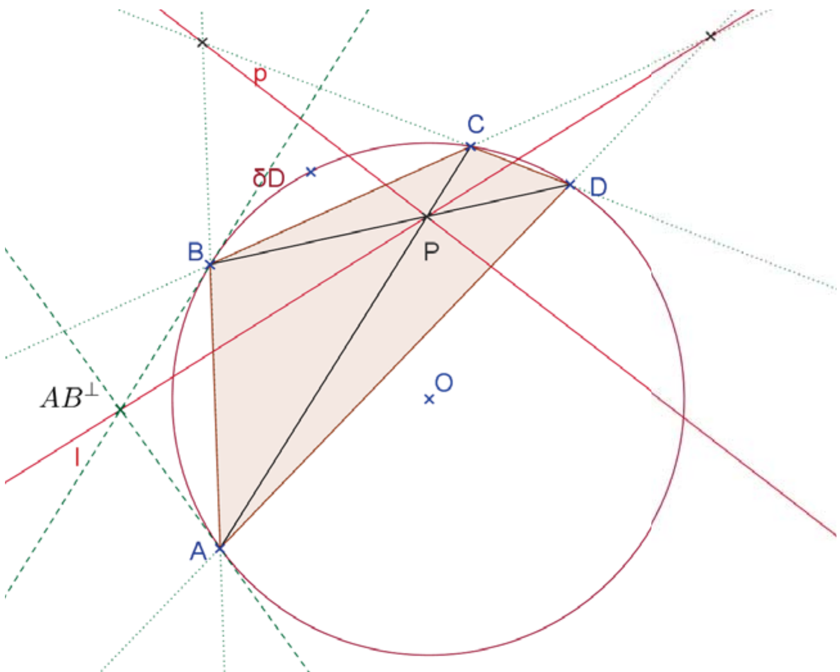

Figure 3.13: An ideal quadrilateral produces five right angles.

shown in the section on the Poincaré model, this special position with $P = O$ can be achieved by means of hyperbolic reflections.

But in that special position, the diagonals and the lines l and p are absolute straight lines, and the ideal quadrilateral $\square ABCD$ appears as a rectangle. The five right angles are immediate to confirm. $\square$

We now use the ideal quadrilateral for the construction of a hyperbolic reflection.

Construction 3.3 (Hyperbolic reflection of a point by a given line). *Given is a reflection line l and a point K not on line l. We want to construct the reflective image of K' of K by the line l.*

Choose any parallel to l with ideal ends B and C, and get the polar $BC^{\perp}$ as intersection point of the tangents at B and C to the circle of infinity ∂D. The line p through the points K and $BC^{\perp}$ is the perpendicular dropped from K onto line l. The foot point P is the intersection of lines p and l.

Finally, to get the reflective image, we draw the line CP with the second end A, and BP with the second end D, thus producing an ideal quadrilateral $\square ABCD$. The reflection point K' is the intersection of side AD with the perpendicular p.

Remark. The four points K, P, the polar $BC^{\perp}$, and the intersection of lines AB and CD lie all on the perpendicular p. This property should be used to achieved better accuracy.

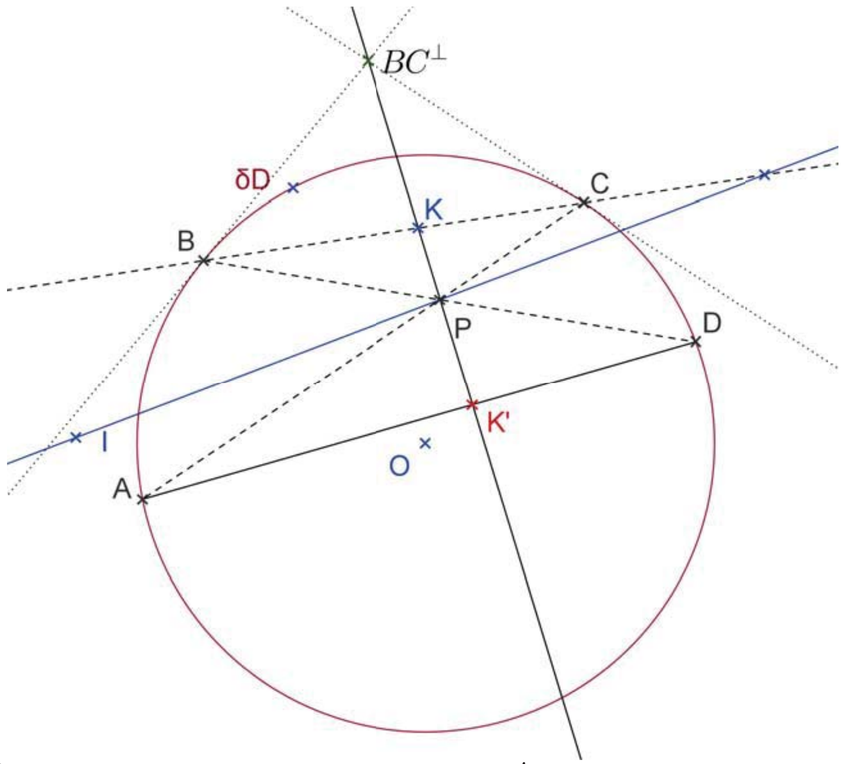

Figure 3.14: Construction of the reflective image K' for a given point K and reflection line l.

Construction 3.4 (**True hyperbolic angle**). *Given is a hyperbolic angle $\angle OKF$. We draw the ends E and F of the line EF and its polar $C = EF^\perp$. The circle around C through the ends E and F intersects the segment OK in point P. We draw segments EB and FA through point P and obtain the ends A and B. These points are actually the endpoints of a diameter. The true hyperbolic angle is $\omega = \angle KOA$.*

Proposition 3.10 (**Distortion of angles**). *The measure of the hyperbolic angle $\angle OKF$ can be obtained by the construction from the figure on page 819 by a reflection across the point P. One obtains the true hyperbolic angle $\omega = \angle KOA$ with vertex at the center. By the formula*

$$\tan\omega = \tan\alpha\sqrt{1-r^2} \tag{3.11}$$

the true angle ω is calculated from the apparent angle α.

Justification of the construction. The mapping from Poincaré's to Klein's model as explained in proposition 3.1 gives

$$|OK| = \frac{2\,|OP|}{1+|OP|^2} \tag{3.1}$$

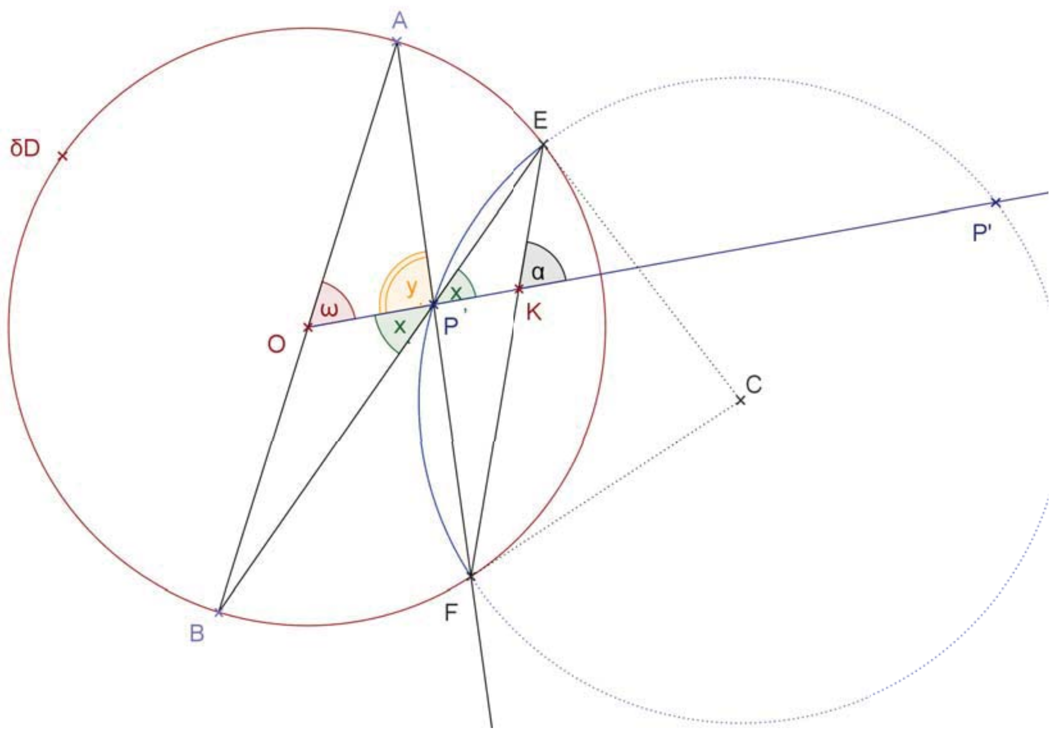

Figure 3.15: Construction of the true hyperbolic angle $\angle OKF$.

With the short-hands $r = |OK|$ and $p = |OP|$ we get

$$r = \frac{2p}{1+p^2} \quad , \quad \frac{1-r}{1+r} = \frac{(1-p)^2}{(1+p)^2} \quad \text{and} \quad \frac{(1-r)(1+p)}{(1+r)(1-p)} = \frac{(1-p)}{(1+p)}$$

Hence point P is the hyperbolic midpoint of segment OK. The hyperbolic point reflection across P maps $K \mapsto O, E \mapsto B, F \mapsto A$. Hence points A, B and O lie on a line, and the angle $\angle PKF$ is reflected to the congruent angle $\angle PKA$, as claimed. □

Justification of the formula (9.21). With the cos theorem for triangles $\triangle PAO$ and $\triangle PBO$ we get

$$|PA|^2 = 1 + p^2 - 2p\cos\omega \quad , \quad |PB|^2 = 1 + p^2 + 2p\cos\omega$$
$$|PA|^2|PB|^2 = (1+p^2)^2 - 4p^2\cos^2\omega = (1-p^2)^2 + 4p^2\sin^2\omega$$

We apply the sin theorem for all four triangles $\triangle PAO, \triangle PBO, \triangle PEK, \triangle PFK$ and get

$$
\begin{aligned}
\frac{\sin^2\omega}{|PA||PB|}\,\frac{|PE||PF|}{\sin^2\alpha} &= \frac{\sin x\sin y}{|OA||OB|}\,\frac{|KE||KF|}{\sin x\sin y}\\
\frac{\sin^2\omega}{\sin^2\alpha} &= \frac{|PA||PB|}{|PE||PF|}\,|KE||KF| = |PA|^2|PB|^2\,\frac{|KE||KF|}{|PA||PB||PE||PF|}\\
&= \left[(1-p^2)^2+4p^2\sin^2\omega\right]\,\frac{(1-r)(1+r)}{(1-p)^2(1+p)^2}
\end{aligned}
$$

Reminding

$$
r=\frac{2p}{1+p^2}\quad,\quad r^2=\frac{4p^2}{(1+p^2)^2}\quad,\quad 1-r^2=\frac{(1-p^2)^2}{(1+p^2)^2}\quad,\quad \frac{r^2}{1-r^2}=\frac{4p^2}{(1-p^2)^2}
$$

we get

$$
\begin{aligned}
\frac{\sin^2\omega}{\sin^2\alpha} &= \left[1+\left(\frac{2p}{1-p^2}\right)^2\sin^2\omega\right](1-r^2) = 1-r^2+r^2\sin^2\omega\\
\sin^2\omega &= \frac{(1-r^2)\sin^2\alpha}{1-r^2\sin^2\alpha}\\
\cos^2\omega &= \frac{1-\sin^2\alpha}{1-r^2\sin^2\alpha}\\
\tan^2\omega &= (1-r^2)\tan^2\alpha
\end{aligned}
$$

Moreover, we see that the angle ω is acute if and only if the angle α is acute, ω is right if and only if α is right, and ω is obtuse if and only if α is obtuse. Hence we get formula (9.21) with the positive square root. □

3.4 Engel's Theorem

In this paragraph, we state and prove Engel's theorem. This theorem constructs a bijective correspondence between right triangles and Lambert quadrilaterals. Klein's model is convenient for the proof, which uses a refinement of Bolyai's construction.

Recall that quadrilaterals with three right angles are called *Lambert quadrilaterals*. Right triangles and Lambert quadrilaterals both have five basic pieces (angles or sides), and can be constructed once any two of them are given.

Proposition 3.11 (Transfer of a segment on its line). *Let E and F be the ideal endpoints of line c. We put any two ideal points A, B on one of the two arcs from E to F, and two more points V, W on the other arc on ∂D from E to F. Let P_2P_3 and Q_2Q_3 be the segments on line l cut out by the angles $\angle VAW$ and $\angle VBW$, respectively.*

The two segments P_2P_3 and Q_2Q_3 are hyperbolically congruent.

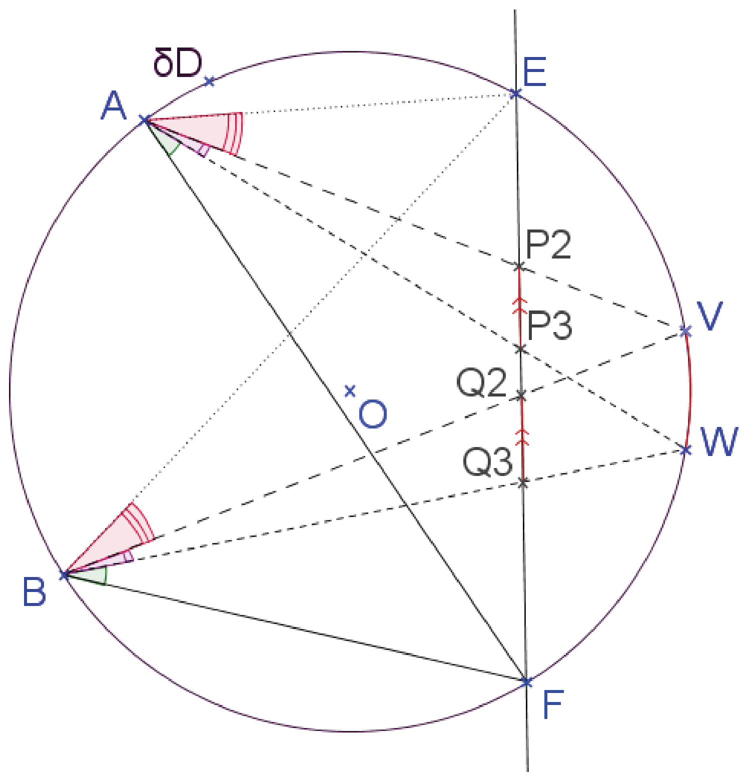

Figure 3.16: Transfer of a segment along its line.

Proof. We need these two basic facts:

(**Invariance of the cross ratio for central projections**). *Given are any two lines across a bundle of four rays with common vertex. Let P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4 be the intersection points of the rays with the two lines across the bundle. Then the cross ratios $(P_1P_2, P_3P_4) = (Q_1Q_2, Q_3Q_4)$ are equal. The same statement holds for two lines intersecting two congruent ray bundles.*

(**Euclid's III.21**). *Two angles from points of a circle subtending the same arc are congruent.*

By Euclid III.21, the two ray bundles $\overrightarrow{A(E,V,W,F)}$ and $\overrightarrow{B(E,V,W,F)}$ are congruent. The two ray bundles intersect line EF in the four points E, P_2, P_3, F and E, Q_2, Q_3, F, respectively. Now we use fact 1 with $E = P_1 = Q_1$ and $F = P_4 = Q_4$ and get

$$(P_2P_3, EF) = (Q_2Q_3, EF) \quad \text{and} \quad s(P_2, P_3) = s(Q_2, Q_3)$$

as to be shown. □

For s given, one defines s^* to be the hyperbolic length for which $\pi(s) + \pi(s^*) = 90°$. The pieces of $\triangle ABC$ are denoted in Euler's standard fashion.

Proposition 3.12 (Engel's Theorem). *There is a bijective (one-to-one and onto) correspondence between right triangles and Lambert quadrilaterals* $\square PQRS$*. The Lambert quadrilateral* $\square PQRS$ *and the right triangle* $\triangle ABC$ *are matched by putting one leg of the Lambert quadrilateral onto one leg of the right triangle, and one of the outer right angles of the Lambert quadrilateral onto the right angle of the triangle. Thus* $A = P$ *and* $C = S$*. The correspondence is established by requiring* (3.13) *and any one of the other four statements* (3.12),(3.14),(3.15) *or* (3.16)*.*

In that way, the five pieces of the triangle and of the Lambert quadrilateral are matching as follows:

(3.12)	$a := BC$	has angle of parallelism	$\pi(a) = \angle QRS$
(3.13)	$b := AC$		$= PS$
(3.14)	$c := AB$		$= QR$
(3.15)	$\alpha := \angle BAC = \pi(l^*)$		with $l = PQ$
(3.16)	$\beta := \angle CBA = \pi(m)$		with $m = RS$

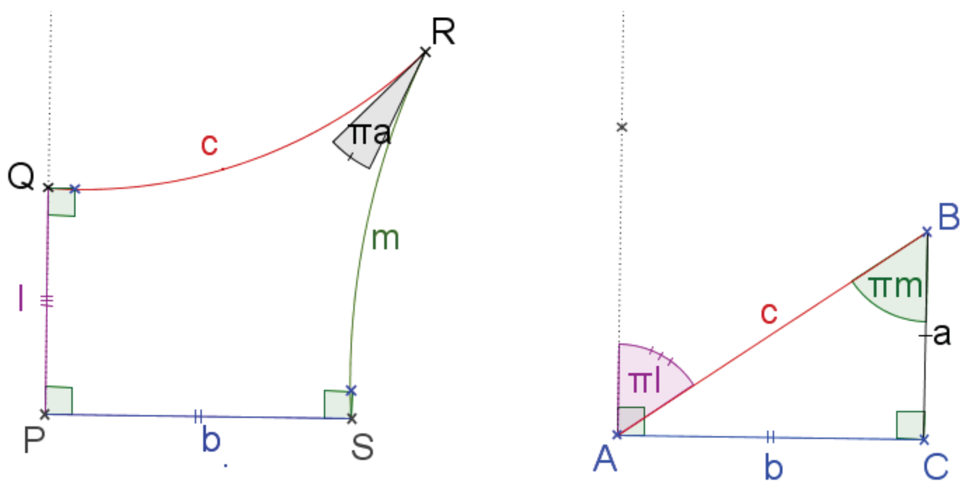

Figure 3.17: The correspondence of a Lambert quadrilateral and a right triangle.

Proof. We use Klein's disk model with disk D and choose as center $O = A = P$. The bijective correspondence of $\triangle ABC$ and $\square PQRS$ is given by requiring that

(i) $A = P$ and $C = S$.

(ii) vertex B lies on the ray $\overrightarrow{SR}$.

(iii) Hypothenuse AB and quadrilateral side QR have a common ideal endpoint V.

In the figure on page 806 about Bolyai's construction of the asymptotic parallel ray, one can see at once that requirements (i)(ii)(iii) define a bijection between right triangles and

Lambert quadrilaterals. For this correspondence, claim (3.13) is obvious. Claim (3.14) and (3.15) both follow from Bolyai's construction. $\square$

Reason of claim (3.15). Indeed point P and line UV have the angle of parallelism $\angle VPQ$, as angle between the perpendicular PQ and the asymptotic parallel ray $\overrightarrow{PV}$. Because $\angle VAQ$ is the complementary angle of $\alpha = \angle CAB$, one gets $\pi(PQ) = 90° - \alpha$ and hence $\alpha = \pi(s(P,Q)^*)$. This confirms claim (3.15). $\square$

Reason of claim (3.12). As shown in figure 3.4, we draw line $V'S$ and let W be its second ideal endpoint. Draw line US and let X be its second ideal endpoint.

The lines RS and PS are perpendicular, and intersect in S. The ideal quadrilateral $\square V'UWX$ has its diagonals intersecting in S. Its side UV' is perpendicular to PS. Hence by proposition 3.9, the opposite side WX is perpendicular to PS, too. Furthermore, the other two opposite sides UW and $V'X$ are perpendicular to RS. Thus the quadrilateral $\square V'UWX$, its diagonals, and the two perpendicular lines RS and PS produce the English flag and five right angles, as shown in figure 3.4.

Now we complete the proof of claim (3.12): Let H be the intersection point of the two perpendicular lines UW and RS. It can happen that H lies inside the segment SB, or inside the segment BR, or $H = B$. The figure 3.4 shows the case that H lies inside the segment BR.

Let E and F be ideal endpoints of line $RS = EF$. As explained in proposition 3.11 about the transfer of a segment on its line, the two ray bundles $\overrightarrow{U(E,V,W,F)}$ and $\overrightarrow{V'(E,V,W,F)}$ are congruent. Hence the two segments cut out on line EF are congruent:

$$(3.17) \qquad (HR, EF) = (SB, EF) \quad \text{and} \quad s(H,R) = s(S,B) = a$$

The angle of parallelism of segment RH is $\angle URH$, because it is the angle between the asymptotic parallel ray $\overrightarrow{RU}$ and the perpendicular RH, for point R and line UW. Hence $\pi(RH) = \angle URH$. Combined with (3.17), we conclude Hence $\pi(a) = \pi(RH) = \angle URH = \angle QRS$ as claimed in (3.12). $\square$

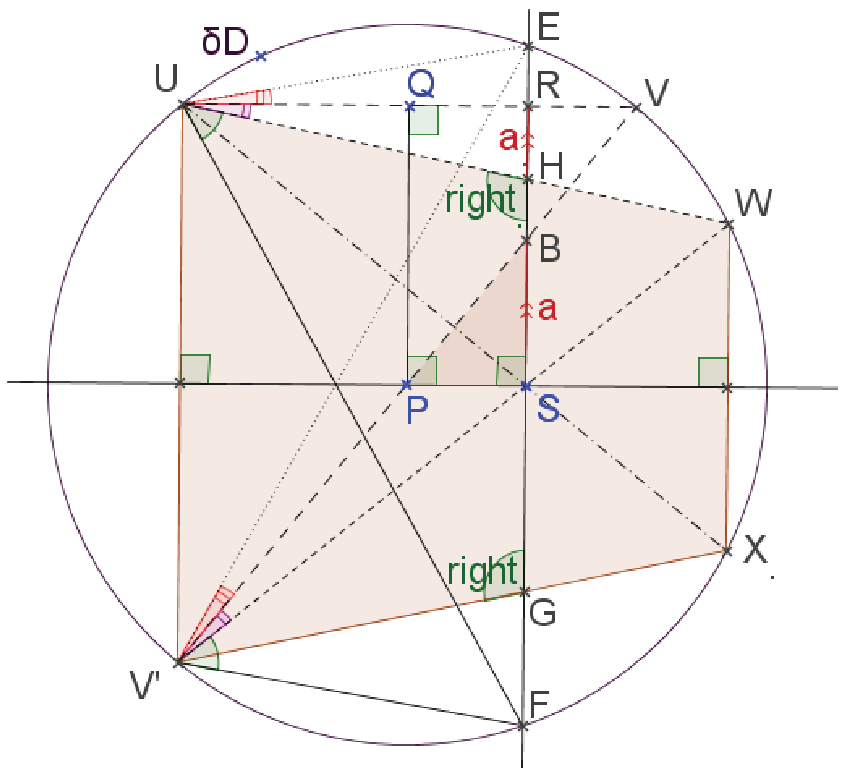

Figure 3.18: Find the English flag.

Reason of claim (3.16). This is done in the same manner we just have proved (3.12). This time, we use line US and let X be its second ideal endpoint. As shown above, the two segments UW and $V'X$ are both perpendicular to RS. Let G be the intersection point of the perpendicular lines $V'X$ and SR.

Again by Euclid III.21, the two ray bundles $\overrightarrow{U(E,V,X,F)}$ and $\overrightarrow{V'(E,V,X,F)}$ are congruent. The two ray bundles intersect line EF in the four points E,R,S,F and E,B,G,F, respectively. Hence proposition 3.11 implies

$$(RS,EF) = (BG,EF) \quad \text{and} \quad s(R,S) = s(B,G) = m \tag{3.18}$$

Referring to point B and line $V'X$, we see that the angle of parallelism of segment BG is $\angle V'BG$, because it is the angle between the asymptotic parallel ray $\overrightarrow{BV'}$ and the perpendicular BG. Hence $\pi(BG) = \angle V'BG$. Combining with (3.18), we conclude that $\pi(m) = \pi(RS) = \pi(BG) = \angle V'BG = \angle ABC = \beta$, as claimed in (3.16). □

3.5 The Hjelmslev quadrilateral

Theorem 3.2. *Hjelmslev's theorem 10.2 about the quadrilateral with right angles at two opposite vertices holds in hyperbolic and neutral geometry, too.*

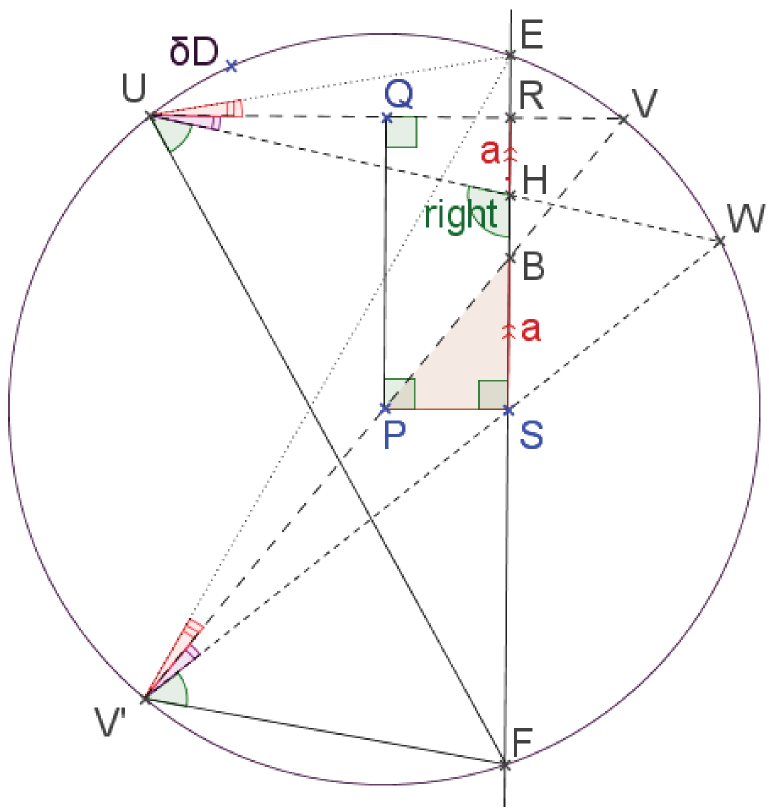

Figure 3.19: Proof of claim (3.12): a has angle of parallelism $\pi(a) = \angle URH$.

Problem 3.5. *Prove the angle congruence of Hjelmslev's Theorem in hyperbolic geometry. Assume that the Euclidean theorem has already been shown. Use Klein's model, and put into the center of the disk the vertex of the pairs of angles that you want to compare.*

Proof of the angle congruence in hyperbolic geometry. I use Klein's model with the vertex B put into the center of the disk. Because angles at the center of the disk are absolute (not distorted), it suffices to compare the Euclidean angles at B. The right angles of the Hjelmslev quadrilateral are absolute right angles, too, since one side of them is a diameter of the Klein disk. Hence the hyperbolic Hjelmslev quadrilateral appears in Klein's model as a Euclidean Hjelmslev quadrilateral, too.

Now we can use the Euclidean version of the theorem, and conclude Euclidean congruence of the angles $\beta := \angle ABD$ and $\beta' = \angle GBC$. Because angles at the center of the disk are not distorted, the Euclidean congruence implies the hyperbolic congruence. $\square$

Problem 3.6 (Open problem). *Prove the segment congruence of Hjelmslev's Theorem in hyperbolic geometry. Assume that the Euclidean theorem has already been shown and use Klein's model.*

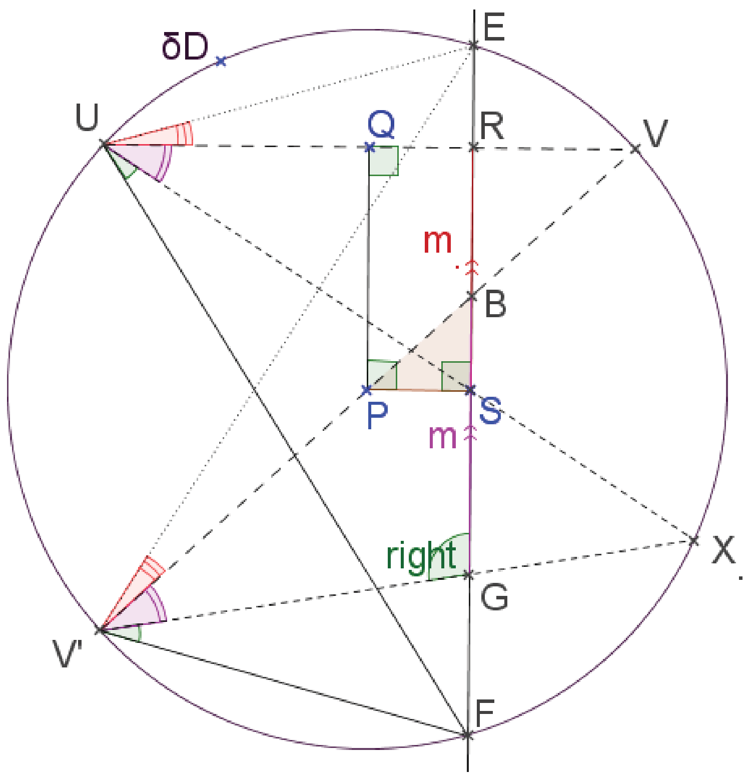

Figure 3.20: Proof of claim (3.16): $m = RS$ has angle of parallelism $\pi(m) = \angle V'BG$.

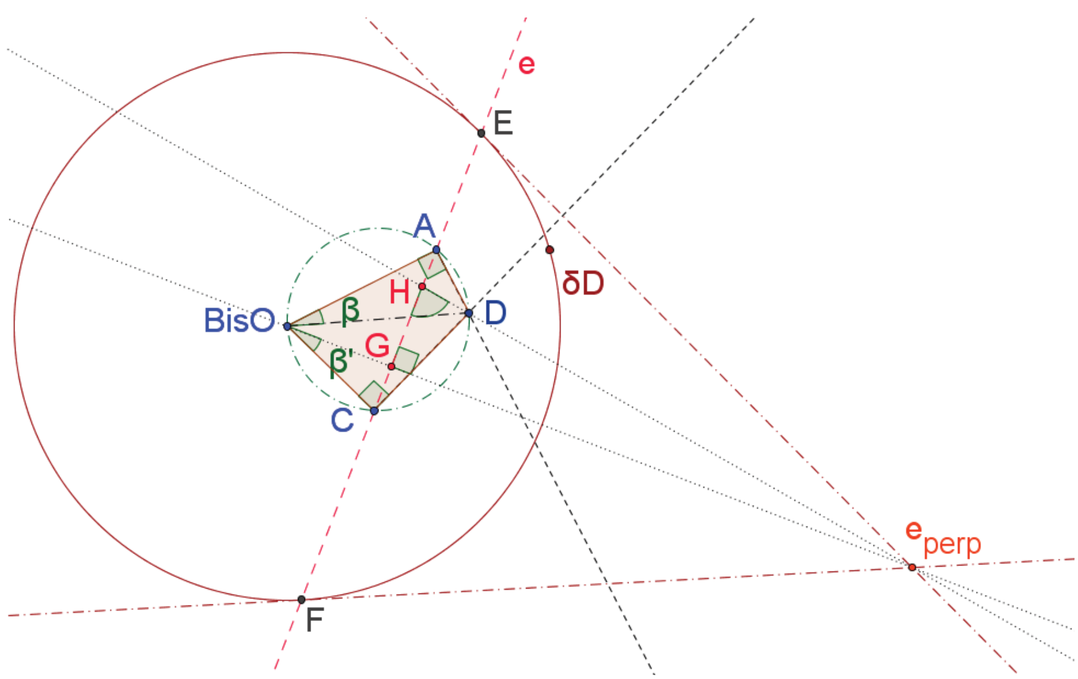

Figure 3.21: The angle congruence of Hjelmslev holds in hyperbolic geometry, too.