8 Hilbert's Axiomatization of Hyperbolic Geometry

The Uniformity Theorem from the section about the natural axiomatization of geometry classifies the Hilbert planes into three basic types. In the definition 10.1 from the section "Towards a Natural Axiomatization of Geometry", we have agreed to call a Hilbert plane, in which the angle sum of every triangle is less than two right angles, a *semi-hyperbolic plane*.

Problem 8.1. *Convince yourself that in a semi-hyperbolic plane, there exist at least two parallels to a given point to a given line. Is the converse true?*

8.1 Some basic facts about semi-hyperbolic planes

Recall that by definition 3.11, two rays are called *coterminal* if one of them can be obtained from the other one by extension or deletion of a segment. Being coterminal defines an equivalence relation among rays.

Definition 8.1 (Limiting parallel ray). A ray k with vertex A is called a *limiting parallel* to ray h with vertex B if these two rays do not intersect, but each ray from vertex A inside the angle $\angle(\overrightarrow{AB}, k)$ intersects the ray h.

It is easy to see: if a ray k is a limiting parallel to ray h, then the same ray k is a limiting parallel to any ray h' coterminal with h, too. We drop the perpendicular from vertex A onto the line of h and obtain the foot point C.

Definition 8.2 (Angle of parallelism). The angle of parallelism is the angle between the limiting parallel ray and the perpendicular from A onto line l. The angle of parallelism depends only on the hyperbolic length (congruence class) of the segment AC.

Problem 8.2. *Convince yourself of the last assertion. Convince yourself that in a semi-hyperbolic plane, the angle of parallelism is always acute, if it exists.*

Be aware that this statement does not imply the existence of the limiting parallel ray.

Definition 8.3. Following Lobachevskij, one defines a special function, called $\pi(s)$, giving the angle of parallelism π for a segment of hyperbolic length s. We shall use the notation

$$\pi^*(s) = 90^\circ - \pi(s)$$

For a semi-hyperbolic plane, these are is indeed partial functions, since existence of the limiting parallel ray is not postulated.

Problem 8.3. *Prove in neutral geometry: Two lines which intersect a transversal with congruent z-angles have a common perpendicular.*

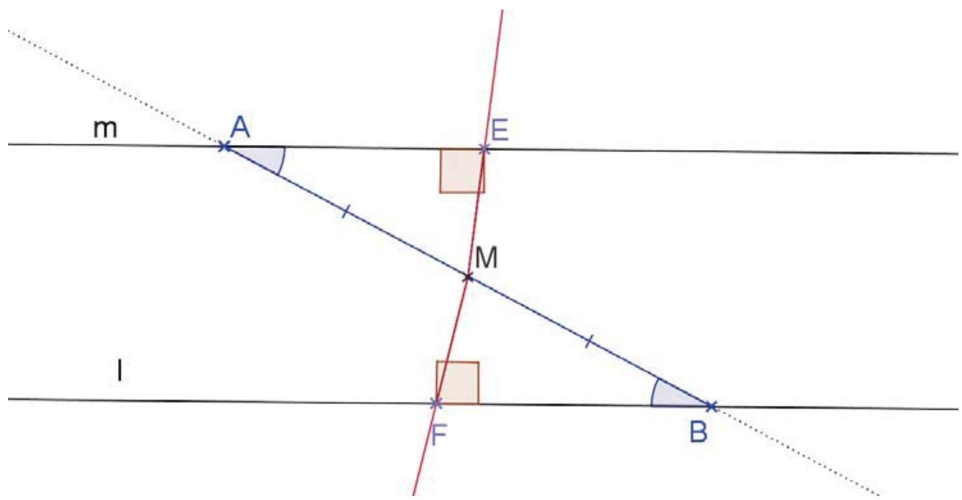

Figure 8.1: Line with congruent z-angles have a common perpendicular.

Reason. Let a transversal line intersect the two (parallel) lines m and l at points A and B, producing congruent z-angles. From the midpoint M of segment AB, we drop the perpendiculars onto the given parallel lines m and l. The foot-points E and F produce the triangles $\triangle MEA$ and $\triangle MFB$—unless the given transversal already is the common perpendicular, in which case we are ready. The two triangles $\triangle MEA \cong \triangle MFB$ are congruent by SAA congruence. Hence we get vertical angles at vertex M. [66] The three points E, F and M lie on a line, which is the common perpendicular to be found. □

8.1.1 The equivalence relation of limiting parallelism

Theorem 8.1. *Being limiting parallel or coterminal defines an equivalence relation among rays. The corresponding equivalence classes are called* ends *or* ideal points.

Let AC be a perpendicular onto line l with foot-point C. Somewhat anticipating such a result, we denote the end of the limiting ray $\overrightarrow{AP}$ by α. Furthermore, we denote the limiting ray with vertex A and end α by $\overrightarrow{A\alpha}$.

Lemma 8.1. *If $\overrightarrow{AP} = \overrightarrow{A\alpha}$ is limiting parallel ray to line l, the "cutted" coterminal ray $\overrightarrow{P\alpha}$ is a limiting parallel to line l, too.*

Proof. Drop the perpendicular from point P onto line l, let F be the foot-point. Take any ray $\overrightarrow{PQ}$ inside the angle $\angle FP\alpha$. The ray $\overrightarrow{AQ}$, from the original vertex A, lies inside the angle $\angle CA\alpha$ and hence intersects line l in a point S. We use Pasch'e axiom with triangle $\triangle ASC$ and line PQ. We conclude that ray $\overrightarrow{PQ}$ intersects segment CS, and hence line l in a point T.

Since Q is an arbitrarily chosen point inside the angle $\angle FP\alpha$, we see that the ray $\overrightarrow{P\alpha}$ is a limiting parallel to line l. □

[66] The illustration is not from Leibniz' "best of all worlds".

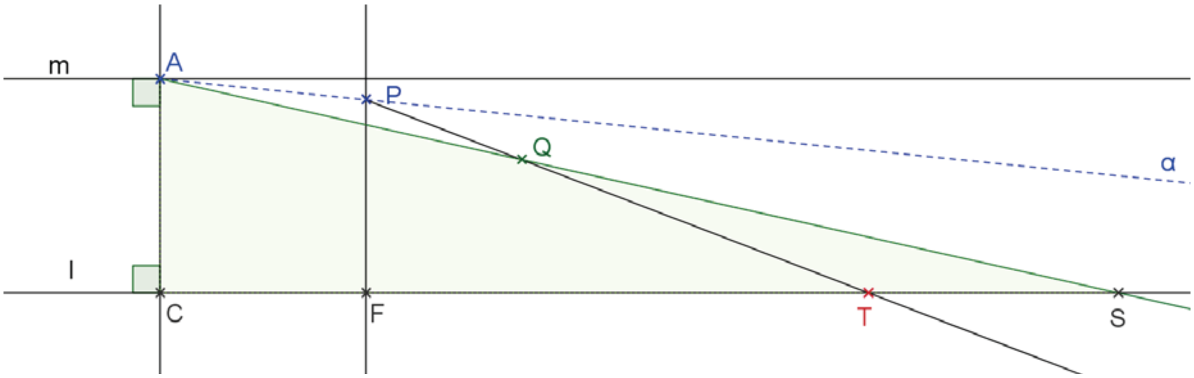

Figure 8.2: Cutting the limiting parallel ray.

Lemma 8.2. *If ray $\overrightarrow{A\alpha}$ is limiting parallel to line l, the coterminally extended ray $\overrightarrow{RA} = \overrightarrow{R\alpha}$ is a limiting parallel to line l, too.*

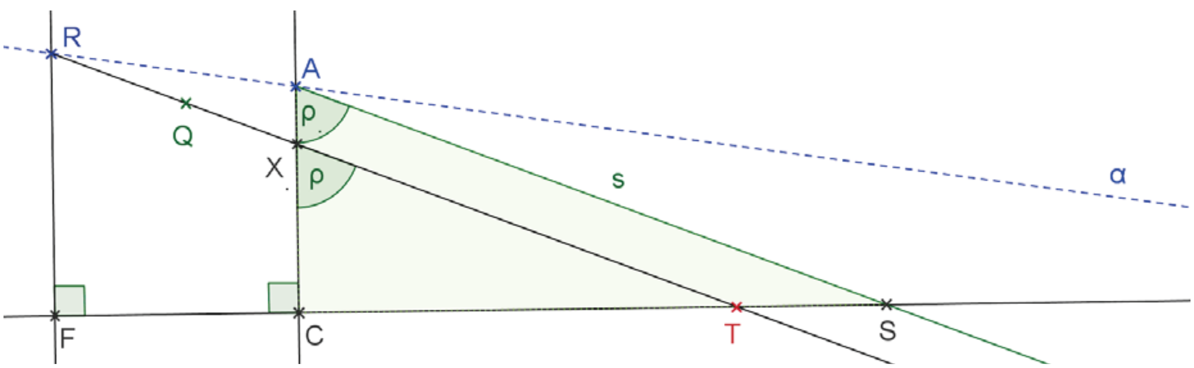

Figure 8.3: Extending a limiting parallel ray.

Proof. Drop the perpendicular from point R onto line l, let F be the foot-point. Take any ray $\overrightarrow{RQ}$ inside the angle $\angle FR\alpha$. If this ray intersects segment FC, we are ready. Otherwise it intersects segment AC, say at point X.

Because of the exterior angle theorem for triangle $\triangle RXA$, we get for the angle of parallelism

$$\angle CA\alpha > \angle AXR = \rho$$

We transfer the angle $\angle AXR = \rho$ onto the perpendicular $\overrightarrow{AC}$ to produce a third ray s. Because of the estimate above, the ray s lies inside the angle $\angle CA\alpha$. Hence it intersects line AC, say at point S.

We use Pasch'e axiom with triangle $\triangle ASC$ and line RQ. We conclude that ray $\overrightarrow{RQ}$ intersects segment CS, and hence line l in a point T.

Since point Q is an arbitrarily chosen point inside the angle $\angle FR\alpha$, we see that the ray $\overrightarrow{R\alpha}$ is a limiting parallel to line l. □

Lemma 8.3. *If a ray k is a limiting parallel to ray h, any ray k' coterminal with k is a limiting parallel to any ray h' coterminal to h.*

Lemma 8.4. *Assume the two lines $l = CB$ and $m = AD$ have the common perpendicular AC. Then the ray $\overrightarrow{AD}$ is not limiting parallel to the ray $\overrightarrow{CB}$.*

This statement holds in any semi-elliptic Hilbert plane, too.

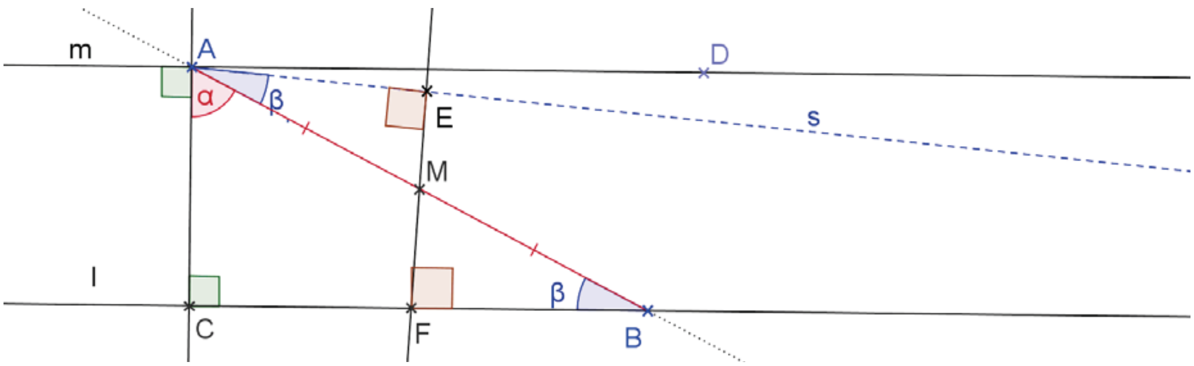

Figure 8.4: Lines with a common perpendicular diverge.

Proof. We consider the case of a semi-hyperbolic plane. Let AC be the common perpendicular. We choose a second point B on line l. Because the geometry is semi-hyperbolic, the triangle $\triangle ABC$ has angle sum less two right angles and hence $\alpha+\beta < R$. We transfer the angle β to vertex A to produce congruent z-angles, and get a new ray $\overrightarrow{AE}$. Since $\alpha + \beta < R$, this ray lies in the interior of the right angle $\angle CAD$ (with side AD lying on m). Ray $\overrightarrow{AE}$ does not intersect line l, because there exists a common perpendicular EF of lines l and AE, as explained in Problem 8.3.

Because ray $\overrightarrow{AE}$ lies in the interior of angle $\angle CAD$ and does not intersect line l, the ray $\overrightarrow{AD}$ is not a limiting parallel to line l. The modification necessary for the

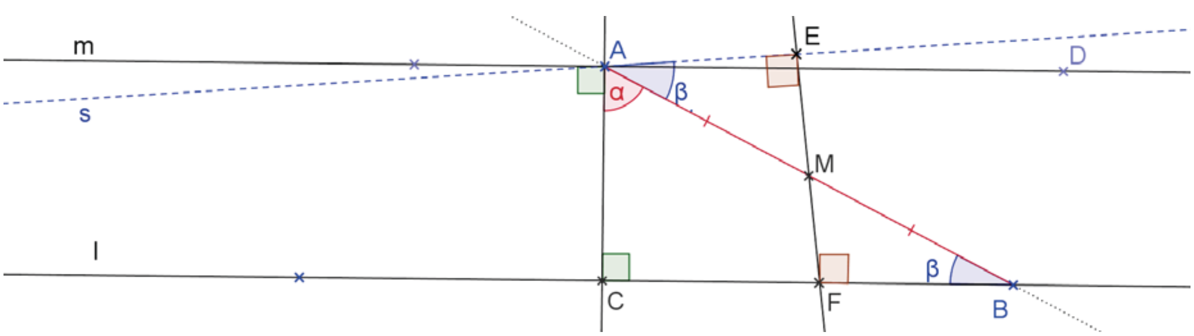

Figure 8.5: Lines with a common perpendicular diverge in the semi-elliptic case, too.

semi-elliptic case is shown in the figure on page 875. □

Lemma 8.5 (Symmetry). *If ray l is limiting parallel to ray k, then ray k is limiting parallel to ray l.*

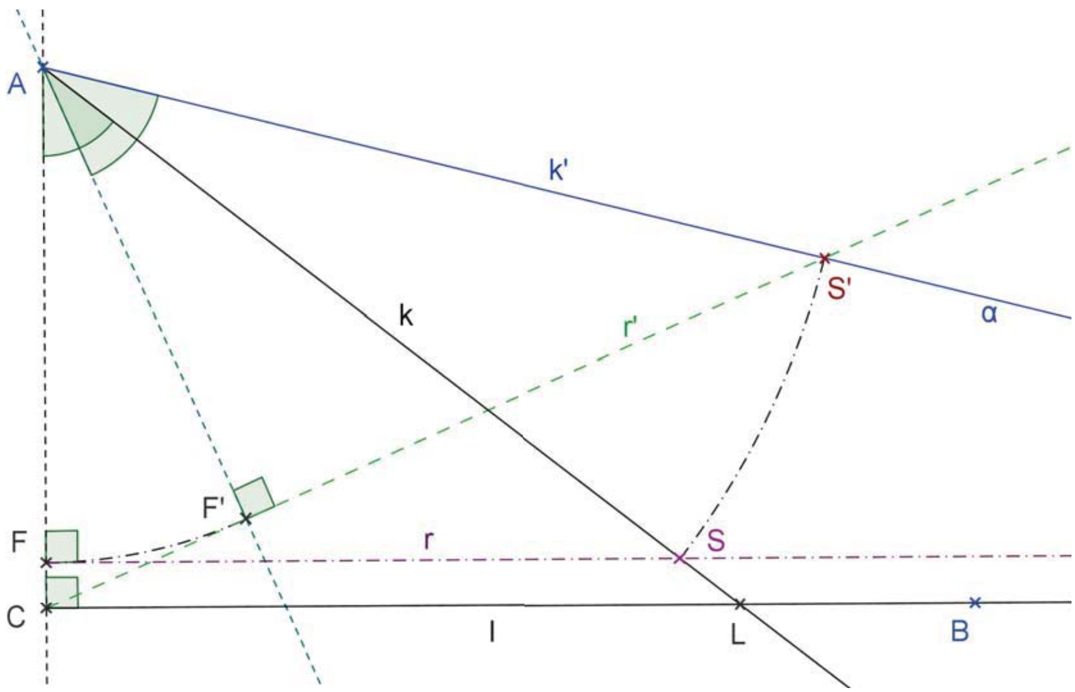

Figure 8.6: If l is limiting parallel to k, then k is limiting parallel to l.

Proof. Let ray k' be a limiting parallel to ray $l = \overrightarrow{CB}$. We may assume that the vertex C of ray l is the foot point of the perpendicular dropped from the vertex A of k' onto l. We want to check that ray l is a limiting parallel to ray k'. To this end, we draw the an arbitrary ray r' inside the angle $\angle BCA$, and check whether it intersects the ray k'.

Let F' be the foot-point of the perpendicular from A onto the ray r'. Since the hypothenuse AC is the longest side of the right triangle $\triangle ACF'$, we can construct a point F between A and C such that $AF \cong AF'$. We produce the congruent angles $\angle FAF' \cong \angle(k, k')$, with the new ray k inside the angle $\angle(\overrightarrow{AC}, k')$.

Since k' is assumed to be a limiting parallel to ray l, the new ray k does intersect l, say in point L. The perpendicular r erected on AC at point F is (divergent) parallel to the ray l. From Pasch's axiom, applied to triangle $\triangle ACL$ and the perpendicular r, we conclude that the ray r does intersect segment AL, say at a point S.

The triangles $\triangle AFS$ and $\triangle AF'S'$ are congruent, actually by ASA congruence. They are obtained from each other by a rotation around point A. The point S' which we have obtained, is the intersection of rays r' and k'.

Since the ray r' can be chosen arbitrarily inside the angle $\angle BCA$, we see that each ray inside this angle intersects the ray k'. Hence the ray l is a limiting parallel to ray k'. □

Lemma 8.6. *Let ray $k = \overrightarrow{A\alpha}$ be a limiting parallel to ray $l = \overrightarrow{CB}$ and the vertex C be the foot point of the perpendicular dropped from the vertex A onto l. The perpendicular r on segment AC, erected at any point F between A and C does intersect the limiting ray k.*

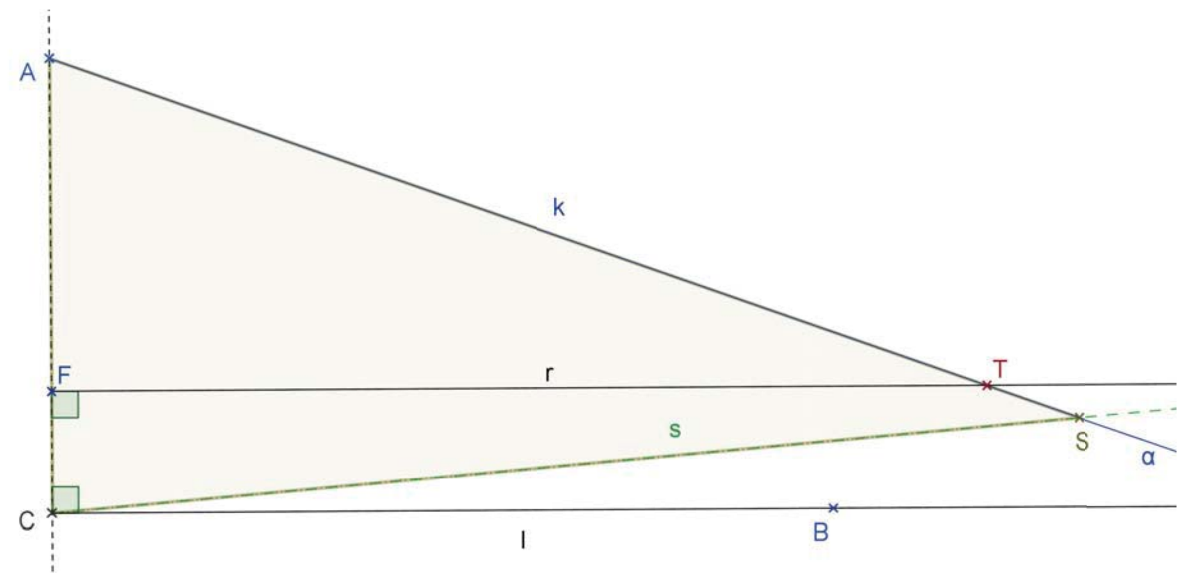

Figure 8.7: Limiting parallel ray k to l comes closer than the ray r.

Proof. By Lemma 8.4, there exists a ray $s = \overrightarrow{CD}$ that remains inside the strip between the divergent parallel rays l and r. By Lemma 8.5, the rays s and k intersect, say at point S. We can now apply Pasch's axiom to triangle ACS and ray r. We conclude that this ray intersects either segment CS or segment AS. The first case is excluded since ray s lies in the half plane of r as its vertex C. Hence we get an intersection point T of segment $AS \subset k$ and ray r. □

Proposition 8.1 (Strict monotonicity of the Lobachevskij function). *If $AB \cong AC$, and a corresponding limiting ray exists, then $\pi^*(AB) = \pi^*(AC)$.*
If $AB < AC$, and the corresponding limiting rays exist, then $\pi^*(AB) < \pi^*(AC)$.

Lemma 8.7. *If two rays are both limiting parallel to a third ray, they are limiting parallel to each other.*

Proof. We assume that the rays k and h are both limiting parallel to a third ray l.

There are two cases to be considered, as shown in the figures on page 878 and 878. Consider the case with two rays k and h lying in different half-planes of a third ray l. By plane separation, we can choose the vertices K, H and L of the three rays lying on a line, with vertex L between K and H.

Take any ray r with vertex K inside the angle $\angle(\overrightarrow{KH}, k)$, and check whether r intersects h. Since ray k is limiting parallel to ray l, the rays r and l intersect in a point S. We can now cut rays r and l to the common vertex S. Because of Lemma 8.5 and Lemma 8.1, the (cutted) ray l is limiting parallel to ray h. Hence the (cutted) ray r intersects ray h. Thus we have checked that rays k and h are limiting parallel.

Secondly, we consider the case assuming the two rays k and h lie in the same half-plane of a third ray l. We can choose the vertices K, H and L of the three rays lying

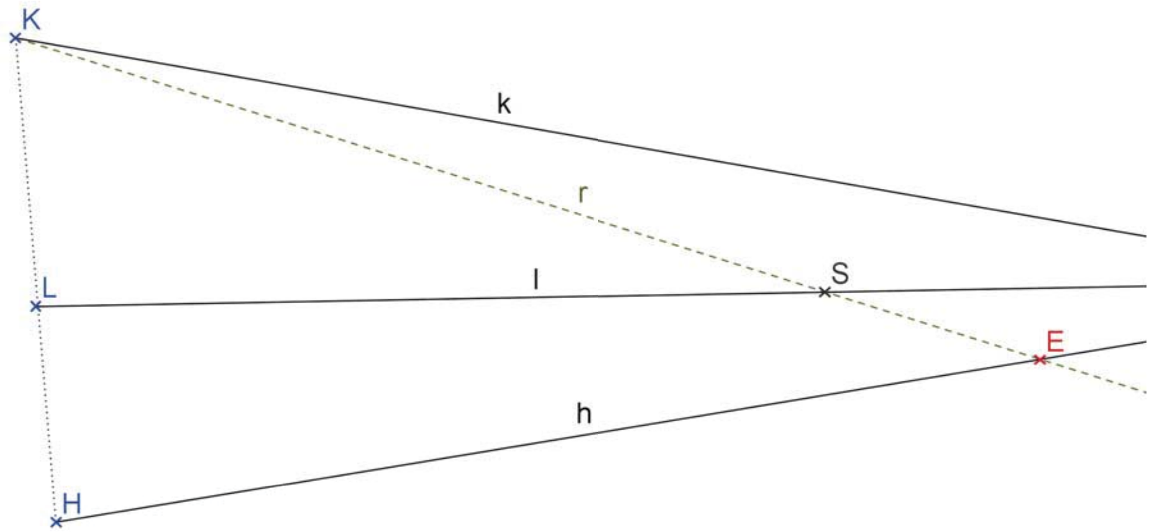

Figure 8.8: Transitivity, case where rays k and h lie in different half-planes of l.

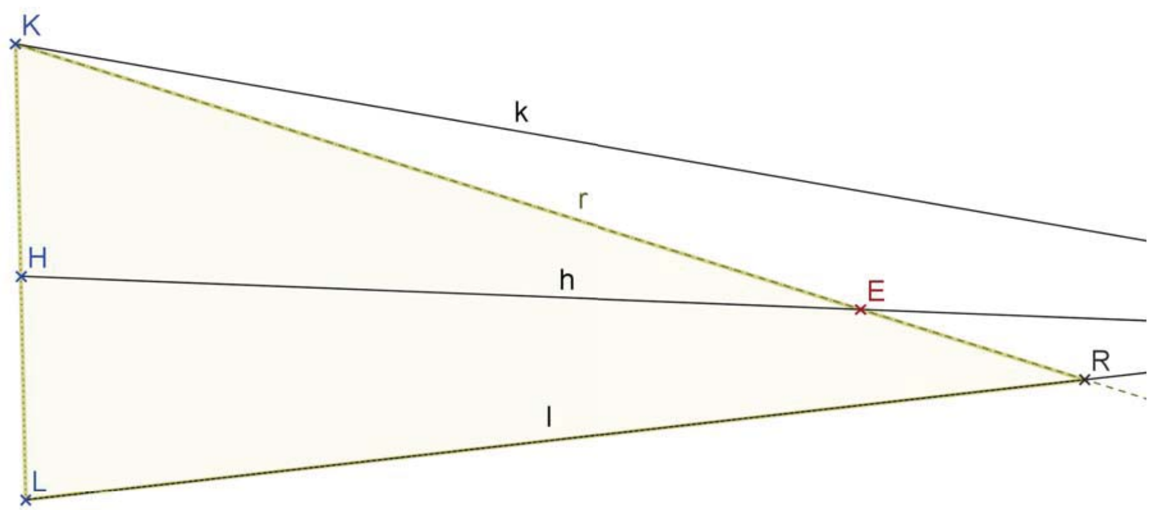

Figure 8.9: Transitivity, case where rays k and h lie in the same half-plane of l.

on a line, with vertex H between K and L. It is enough to check whether ray k is a limiting parallel to ray h.

Consider any ray r with vertex K inside the angle $\angle(\overrightarrow{KH}, k)$, and check whether r intersects h. Since ray k is limiting parallel to ray l, the rays r and l intersect in a point R. We use Pasch's axiom for the triangle $\triangle KLR$ and the line of h. We conclude that the ray h intersects segment $KR \subset r$. Thus we have checked that rays k and h are limiting parallel. □

8.1.2 Limiting triangles

From two ray $\overrightarrow{A\gamma}$ which is a limiting parallel to ray $\overrightarrow{B\gamma}$, we get a *limiting triangle* with two proper vertices A and B and the end γ as third vertex. By definition, it is assumed that the vertices A, B and γ of a limiting triangle do not line on a line.

Proposition 8.2 (SAL-Congruence Theorem). *Assume two limiting triangles have a pair of congruent sides, one pair of congruent angles adjacent to these sides. Then the two triangles are congruent.*

Hint . This fact is a direct consequence of the extended ASA-Theorem 5.11, already explained in the section on neutral triangle geometry, together with Lemma 8.5. □

Proposition 8.3 (ASAL-Congruence Theorem). *Given is a limiting triangle and a segment congruent to its proper side. The two angles at the vertices of this side are transferred to the endpoints of the segment, and reproduced in the same half plane. Then the newly constructed rays are limiting parallel. Hence one gets a second congruent limiting triangle.*

Hint . This fact is a consequence of the extended ASA-Theorem 5.11, already explained in the section on neutral triangle geometry. □

From Lemma 8.4 and Lemma 8.3, and finally problem 8.3, we conclude

Proposition 8.4 (Hilbert III.1). *On two lines with a common perpendicular do not lie limiting parallel rays. On two lines which transversed by a third one with congruent z-angles do not lie limiting parallel rays.*

Proposition 8.5 (Limiting exterior angle theorem). *For a limiting triangle, an exterior angle is greater than the nonadjacent interior angle. Hence the sum of the two interior angles of a limiting triangle is less than two right angles.*

Proof. Let $S(\alpha)$ denote the exterior angle supplementary to angle α. The last proposition 8.4 excludes the case that $S(\alpha) = \beta$.

We assume $S(\alpha) < \beta$ towards a contradiction. Choose a point D on the ray opposite to $\overrightarrow{A\gamma}$ and produce the angle $\angle DAE \cong \beta$. If $\beta > S(\alpha)$, the newly produced ray $\overrightarrow{AE}$ lies inside the angle $\angle BA\gamma$. Since ray $\overrightarrow{A\gamma}$ is a limiting parallel to ray $\overrightarrow{B\gamma}$, the newly produced ray $\overrightarrow{AE}$ intersects ray $\overrightarrow{B\gamma}$, say at point C. Now the exterior angle theorem is violated for triangle $\triangle ABC$. Thus the case $S(\alpha) < \beta$ is ruled out. Hence $S(\alpha) > \beta$ is the only possibility left. □

Remark. By proposition 5.27, taking supplements reverses the order. We conclude that $\alpha < S(\beta)$, which is no contradiction.

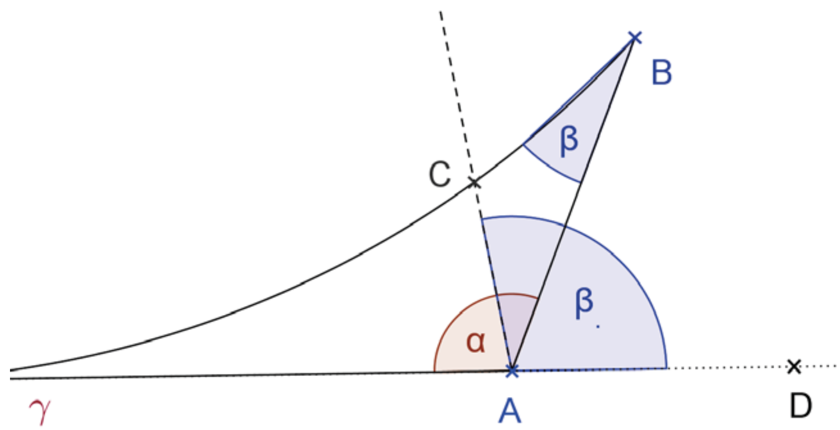

Figure 8.10: The exterior angle theorem for a limiting triangle; with the two congruent angles β, the case drawn is impossible.

Proposition 8.6 (**AAL-Congruence Theorem**). *Assume two limiting triangles have two pairs of congruent angles adjacent to their proper side. Then the two triangles are congruent.*

Hint . This is consequence of Lemma 8.3 about z-angles and strict monotonicity of the Lobachevskij function (Proposition 8.1). □

8.2 The hyperbolic parallel postulate

Definition 8.4 (**Hilbert's Hyperbolic Parallel Postulate**). Given a line l and a point P not on l, there exist two rays r_+, r_- from vertex P which do not lie on the same line and do not intersect the line l, but nevertheless any ray from the vertex P in the interior of the angle $\angle(r_+, r_-)$ intersects the line l.

Proposition 8.7. *In a semi-elliptic plane do not exist any limiting parallel rays. Hence a plane satisfying Hilbert's Hyperbolic Parallel Postulate is semi-hyperbolic.*

Proof. The arguments leading to the exterior angle theorem 8.5 for the limiting triangle even hold in the semi-elliptic case. [67]

Assume towards a contradiction that a semi-elliptic plane contains a ray $\overrightarrow{A\gamma}$ which is limiting parallel to a line l. We drop the perpendicular from point A onto line l to get the foot-point F. Let $\pi(AF) = \angle FA\gamma$ be the angle of parallelism.

It is immediately possible to construct a Saccheri quadrilateral $\square ABFG$, with point B on the same side of perpendicular AF as the ray $\overrightarrow{A\gamma}$. Let $\delta = \angle FAB \cong \angle GBA$ be

[67] It is not obvious at this point that we refer to a truth about a nonexisting object

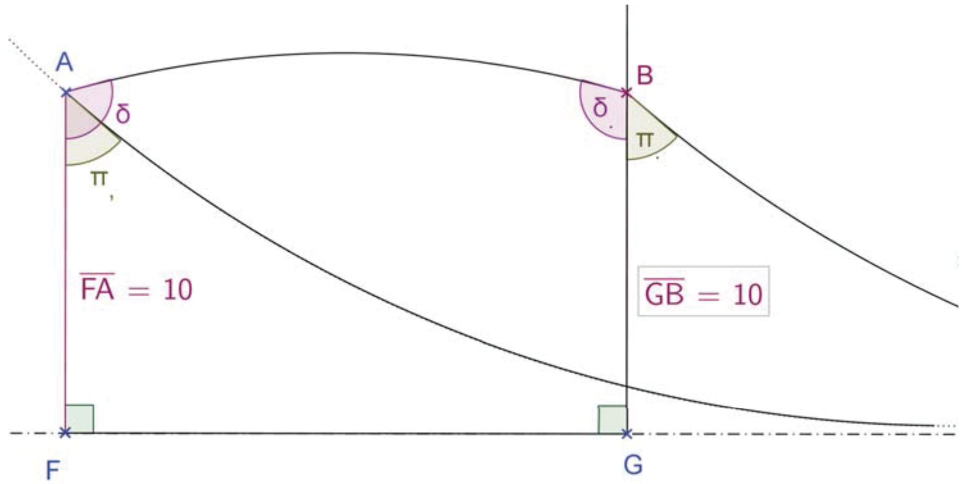

Figure 8.11: There exist no limiting parallel rays in the semi-elliptic case.

the top angle of this Saccheri quadrilateral. Too, we can construct a limiting triangle $\triangle AB\gamma$. It has the angles

$$\angle \gamma AB = \delta - \pi(AF) \quad \text{and} \quad \angle AB\gamma = \delta + \pi(AF)$$

Hence the exterior angle theorem 8.5 implies

$$2R - \delta + \pi(AF) > \delta + \pi(AF)$$

We conclude that $\delta < R$ and hence the plane is semi-hyperbolic. □

Definition 8.5 (Hyperbolic Hilbert plane). A *hyperbolic plane* is a semi-hyperbolic Hilbert plane for which Hilbert's Hyperbolic Parallel Postulate holds.

Remark. I repeat the assumption that the plane is semi-hyperbolic, since it is not obvious at all that no limiting rays exist in a semi-elliptic Hilbert plane.

8.3 Construction of the common perpendicular

Proposition 8.8 (Hilbert III.2). *For any two lines l and m which neither intersect nor have limiting parallel rays, there exists a common perpendicular.*

Construction 8.1 (Construction of the common perpendicular). *Given are any two lines l and m which neither intersect nor have limiting parallel rays.*

We choose two points A and C on the line l and drop the perpendiculars onto line m, with the foot-points B and D. If $CD \cong AB$, we get the Saccheri quadrilateral $\square ABDC$ and proceed to the last step below.

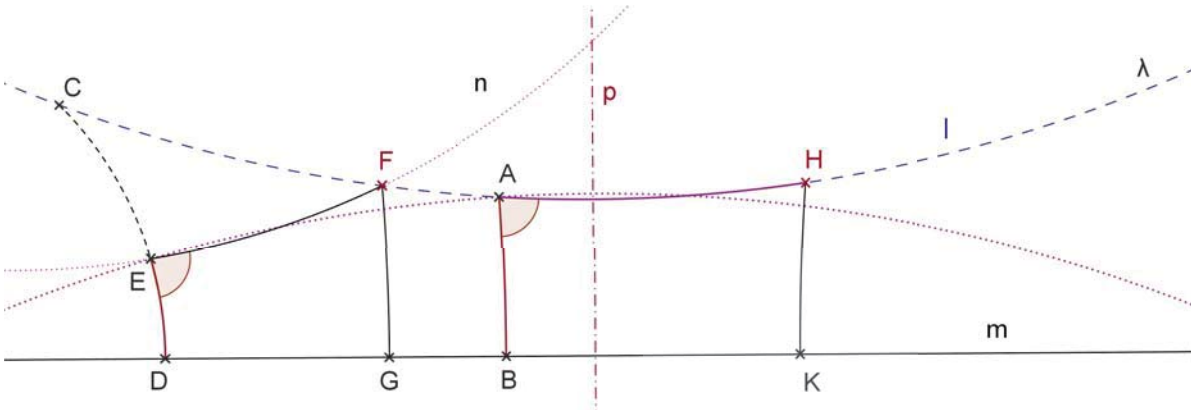

Figure 8.12: Hilbert's construction of the common perpendicular, case $\chi > \varepsilon$.

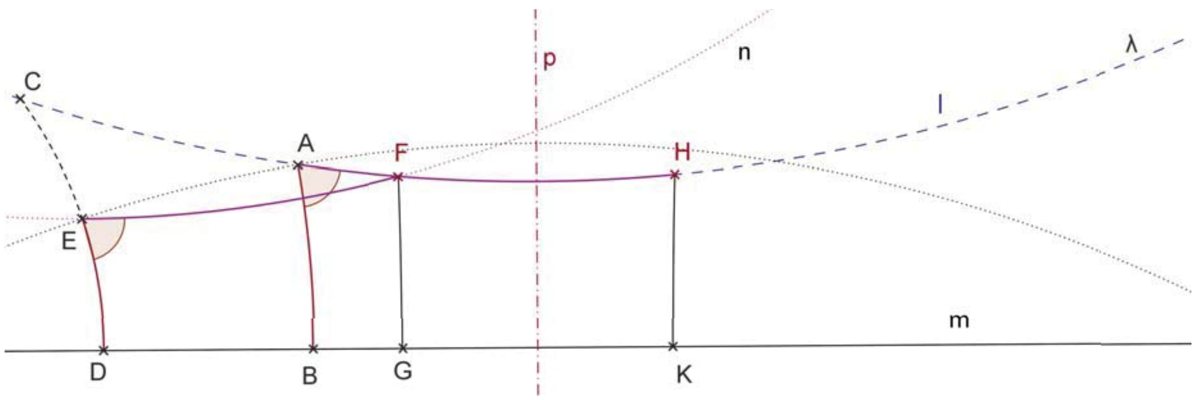

Figure 8.13: Hilbert's construction of the common perpendicular, case $\chi < \varepsilon$.

We may assume that $CD > AB$. We produce the point E between C and D such that $DE \cong BA$. Let λ be the end of ray $\overrightarrow{CA}$. The exterior angle $\chi = \angle BA\lambda$ of the quadrilateral $\square ABDC$ at vertex A is transferred onto the ray $\overrightarrow{ED}$ to produce a new ray n, in the same half-plane of ED as ray $\overrightarrow{C\lambda}$.

There exists an intersection point F of the ray $\overrightarrow{C\lambda}$ with the ray n. Next the segment EF is transferred onto the ray $\overrightarrow{A\lambda}$ to produce congruent segments $EF \cong AH$. We drop the perpendiculars from points F and H and get on the line m the foot-points G and K.

We have obtained a Saccheri quadrilateral $\square FGKH$. The perpendicular bisector p of segment GK is the common perpendicular of the two lines l and m.

Lemma 8.8. *There exists an intersection point F of the ray $\overrightarrow{C\lambda}$ with the ray n, and hence $\angle DEF \cong \angle BA\lambda$.*

Let $\varepsilon = \angle DEA$ be the top angle of the Saccheri quadrilateral $\square DEAB$ and $\chi = \angle BA\lambda$ be the transferred angle. There are two cases, and a borderline case:

$\chi > \varepsilon$**:** *The intersection F lies in the segment CA, as drawn in figure on page 882.*

$\chi = \varepsilon$**:** *The intersection is $F = A$.*

$\chi < \varepsilon$: *The intersection F lies on the extension ray $\overrightarrow{A\lambda}$, as drawn in figure on page 882. In this case $EF > AF$ since the opposite angles in triangle $\triangle AEF$ are $\varepsilon + \chi > \varepsilon - \chi$. and across the greater angle lies the longer side.*

It can happen that $F = A$.

Proof of Lemma 8.8. λ is the end of ray $\overrightarrow{CA}$. Let μ denote the end of ray $\overrightarrow{DB}$ and ν the end of ray n. A simple SAL-congruence (see Proposition 8.2) for limiting triangles shows that

$$\triangle\lambda BA \cong \triangle\nu DE \quad \text{hence} \quad \angle\lambda BA \cong \angle\nu DE$$

Because of the assumption that the two given lines do not contains asymptotic rays, we know that $\mu \neq \lambda$. Hence the limiting triangle $\triangle DB\lambda$ exists. The exterior angle theorem for this limiting triangle yields

$$\angle\lambda B\mu > \angle\lambda DB$$

For the complementary angle we get the reversed inequality.

$$\angle\lambda BA < \angle\lambda DE$$

Together with the congruence we conclude

$$\angle\nu DE < \angle\lambda DE$$

The last inequality shows the ray $\overrightarrow{D\nu}$ lies in the interior of angle $\angle\lambda DE = \angle\lambda DC$. Since $\triangle\lambda DC$ is a limiting triangle, we conclude that ray $\overrightarrow{D\nu}$ intersects ray $\overrightarrow{C\lambda}$, say in point T.

Now we use Pasch's axiom for triangle $\triangle CDT$ and the line of ray $n = \overrightarrow{E\nu}$. The side CD is intersected at point E, but the side $DT \subset D\nu$ cannot be intersected. We see that ray n intersects the segment CT in some point F. □

Justification of the construction 8.1. The goal is to produce a Saccheri quadrilateral. As explained in the construction 8.1, we have produced two quadrilaterals $\square DEFG \cong \square BAHK$. It is straightforward to check that they are congruent. Nevertheless, there are several possible cases for the figure obtained. The two congruent quadrilateral may either overlap or not overlap, as seen in the figures on page 882 for case $\chi < \varepsilon$; and page 882 for the case $\chi > \varepsilon$, respectively.

It can happen that $F = A$. But it cannot happen that $F = H$. We rule out this coincidence as follows: Indeed, if $\chi \leq \varepsilon$ this is impossible since F lies inside the segment AC, but point H not, and hence $F * A * H$. If $\chi > \varepsilon$, then $AF < EF \cong HF$ and hence $A * F * H$.

Hence $\square FGKH$ is a Saccheri quadrilateral. The perpendicular bisector p of segment GK is the perpendicular bisector of segment FH, too. Further remaining details are given by Proposition 7.2 (Hilbert's Proposition 36), in the section on Legendre's theorems. Indeed p is the common perpendicular of the two lines l and m. □

8.4 The enclosing line

Proposition 8.9. *Given any angle, there exists a unique line the opposite rays of which are limiting parallels to the two sides of the angles.*

A bid more generally, we easily conclude:

Proposition 8.10 (Hilbert III.3). *For any two rays which are not limiting parallel to each other, there exists a unique line the opposite rays of which are limiting parallels to the two given rays.*

We call this line the *enclosing line* of the angle, or the two given rays.

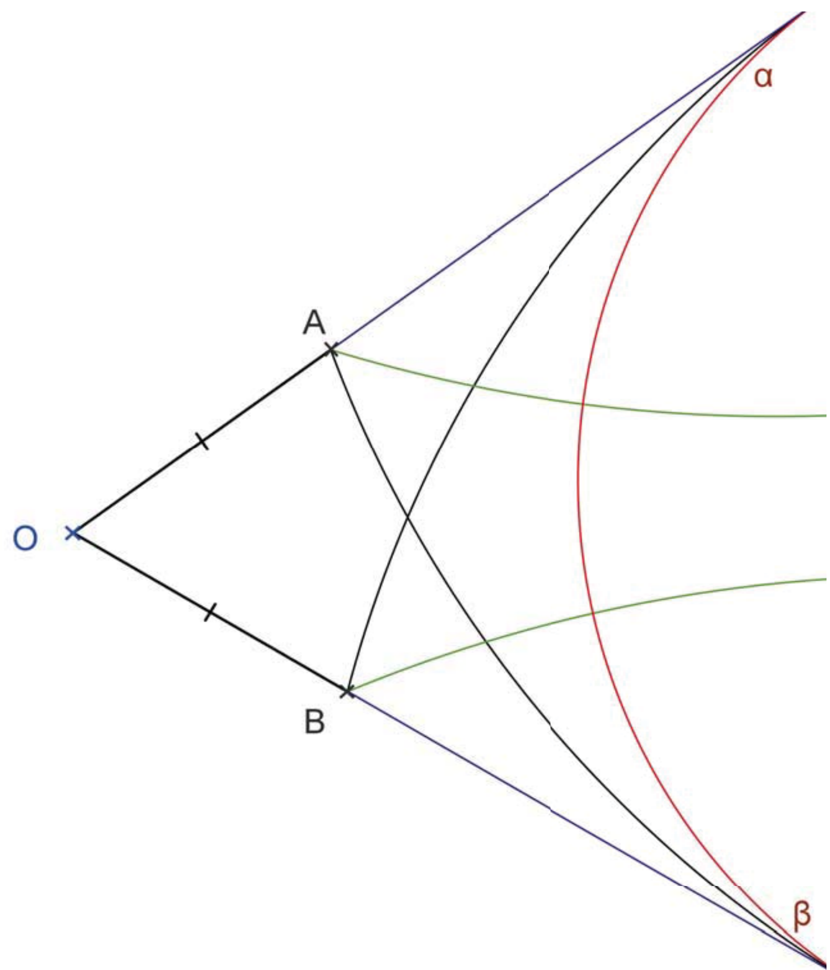

Figure 8.14: Hilbert's construction of the enclosing line.

Construction 8.2 (Construction of the enclosing line). *Given is an angle $\angle \alpha O \beta$ with two different ends $\alpha \neq \beta$. We choose points A and B on the two sides of the angle such that $OA \cong OB$. We draw the limiting rays $A\beta$ and $B\alpha$. Next we construct the angular bisectors s and t of the angles $\angle \alpha A \beta$ and $\angle \beta B \alpha$. The common perpendicular of these two bisectors is the enclosing line with ends α and β.*

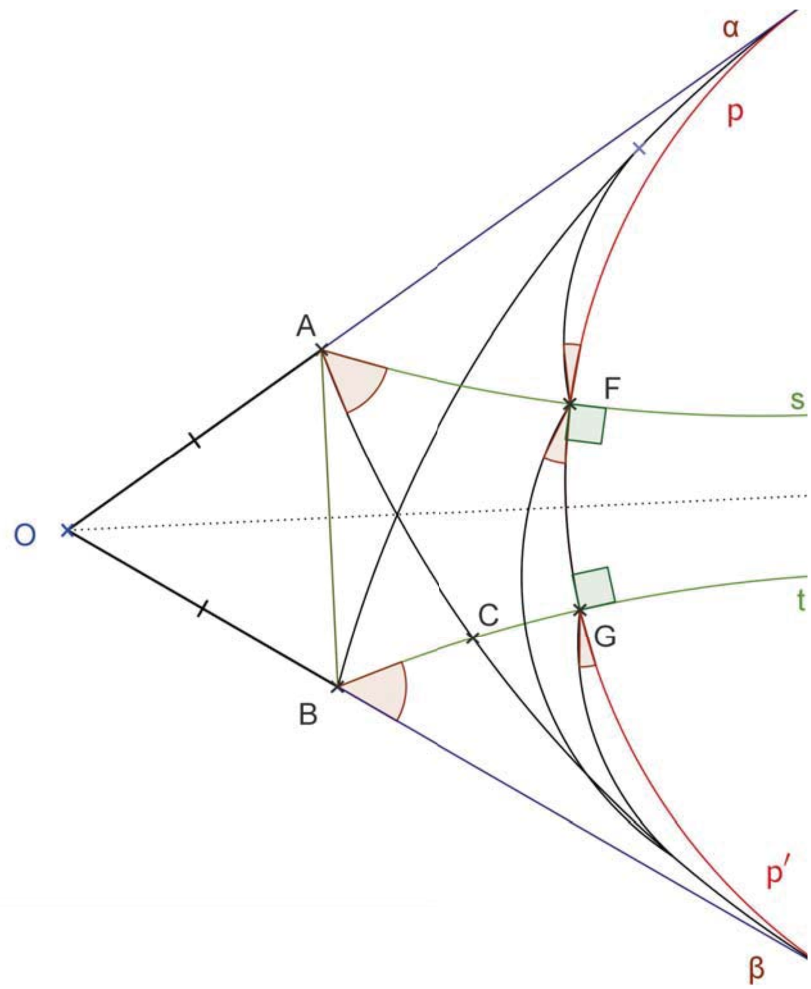

Figure 8.15: Justification for the construction of the enclosing line.

Proof of proposition 8.9. Let s be the end of the angular bisector at vertex A and t be the end of the angular bisector at vertex B. The ray $\overrightarrow{A\beta}$ and the angular bisector t intersect in point C, since by the symmetry Lemma 8.5, the ray $\overrightarrow{B\beta}$ is a limiting parallel to ray $\overrightarrow{A\beta}$. By construction, we get congruent angles

$$\angle CAs \cong \angle CB\beta$$

as indicated in the drawing on page 885. Furthermore, at vertex C there are congruent vertical angles

$$\angle ACt \cong \angle BC\beta$$

Since across the larger angle lies the longer side, we see that $CA > CB$. We conclude from the ASAL-Congruence Theorem 8.3 that the rays $\overrightarrow{As}$ and $\overrightarrow{Ct}$ cannot be limiting parallel.

Neither can they intersect—that would produce a triangle $\triangle ACQ$ congruent to the existing limiting triangle $\triangle BC\beta$ what is impossible. Actually we would have the situation considered in Problem 5.7, and hence again the rays $\overrightarrow{AC}$ and $\overrightarrow{B\beta}$ would intersect, what is impossible.

We see that the lines s and t neither intersect nor contain limiting parallels. Hence, by Proposition 8.8 (Hilbert III.2), the lines of s and t have a common perpendicular.

We have still to check that the common perpendicular has the ends α and β. Denote the ends of the common perpendicular by p and p'. Let F and G be the points on s and t were the common perpendicular intersects them. We draw the three limiting rays $\overrightarrow{F\alpha}$, $\overrightarrow{F\beta}$, and $\overrightarrow{G\beta}$. We get the congruence

$$\angle \alpha Fp \cong \angle \beta Gp'$$

because of the axial symmetry across the bisector of $\angle AOB$. We get the congruence

$$\angle \alpha Fp \cong \angle \beta Fp'$$

because of the axial symmetry across the axis As. Hence $\angle \beta Gp' \cong \angle \beta Fp'$.

If p' would not be limiting parallel to β, a limiting triangle $\triangle FG\beta$ would exist. Hence the exterior angle theorem 8.5 would imply $\angle \beta Gp' > \angle \beta FG = \angle \beta Fp'$. This would contradict the congruence obtained above. Hence the common perpendicular's end p' is β, and similarly the other end p is α, as to be shown. $\square$

Problem 8.4. *Use the enclosing line to prove that Lobachevskij's function $\pi(s)$ assumes all acute angles, for a hyperbolic plan.*

Solution. Given any acute angle α, let $\angle AOB = 2\alpha$ and construct the enclosing line $\alpha\beta$. Let point F be the intersection of the enclosing line with angle bisector of $\angle AOB$. For point O and the enclosing line, the angle of parallelism is the given angle $\angle AOF = \alpha$. $\square$

Proposition 8.11. *Lobachevskij's function $\pi(s)$ is a strictly decreasing bijection from the lengths s of segments to the acute angles.*

8.5 The three reflections Theorem

In this subsection, we refer repeatedly to the section on neutral triangle geometry. Let S_a, S_b, S_c denote the reflections across the side bisectors of the sides a, b, c, respectively, of triangle $\triangle ABC$.

Proposition 8.12 (The three reflections theorem for a triangle with a circum-circle). *Assume that triangle $\triangle ABC$ has a circum-circle and let O be its center. Let S_A, S_B, S_C denote the reflections across the lines OA, OB, OC, respectively. The compositions of these reflections satisfy*

$$S_a \circ S_b \circ S_c = S_B$$

Similarly, we get the relations

$$
\begin{aligned}
S_A &= S_c \circ S_a \circ S_b = S_b \circ S_a \circ S_c \\
S_B &= S_a \circ S_b \circ S_c = S_c \circ S_b \circ S_a \\
S_C &= S_b \circ S_c \circ S_a = S_a \circ S_c \circ S_b \\
S_a &= S_c \circ S_A \circ S_b = S_b \circ S_A \circ S_c \\
S_b &= S_a \circ S_B \circ S_c = S_c \circ S_B \circ S_a \\
S_c &= S_b \circ S_C \circ S_a = S_a \circ S_C \circ S_b
\end{aligned} \tag{8.1}
$$

Proof. As already stated in Proposition 9.2 from the section about neutral triangle geometry, the three side bisectors intersect at the center O of the circum-circle—the existence of which has been assumed. The composite mapping

$$M := S_a \circ S_b \circ S_c \circ S_B$$

maps both points O and B to themselves, mapping the latter point via $B \mapsto B \mapsto A \mapsto C \mapsto B$. Since the mapping M is an isometry, it maps all points of line OB to themselves. Because of preservation of orientation, the mapping cannot exchange the two half plane of line OB. Hence the mapping M is the identity. Any reflection S is an involution, which means $S \circ S$ is the identity. The remaining claims follow easily from this fact. □

Corollary 64. *The product $S_a \circ S_b \circ S_c$ of any three reflections across axes which intersect in a common point is a reflection across a forth axis through this point.*

Problem 8.5. *In a semi-Euclidean plane, three lines a, b and c intersecting in one point are given. Construct a triangle for which a, b and c are the side bisectors. Prove that there exist exactly two such triangles with a given circum-circle.*

Solution of Problem 8.5. Take any point P on the side bisector b and construct its reflection images P_a and P_c across lines a and c, respectively. The line OB is the perpendicular bisector of P_aP_c. We can choose any point B on this line as one of the vertices of the triangle, and get the other vertices A and C by reflections across the given side bisectors c and a, respectively. □

Question. Explain the reason for this construction.

Answer. The mapping $S_B = S_a \circ S_b \circ S_c$ maps point P_c via $P_c \mapsto P \mapsto P \mapsto P_a$. Hence points P_c and P_a are reflection images across the axis of S_B. Vertex B lies on this axis.

We have explained in Theorem 9.5 that not every triangle has a circum-circle. Indeed this Theorem recasts the almost tragical errors of Farkas Bolyai in the present day context of neutral geometry.

Question. Why do I consider the errors of Farkas Bolyai almost tragical. Find an impressive historic source.

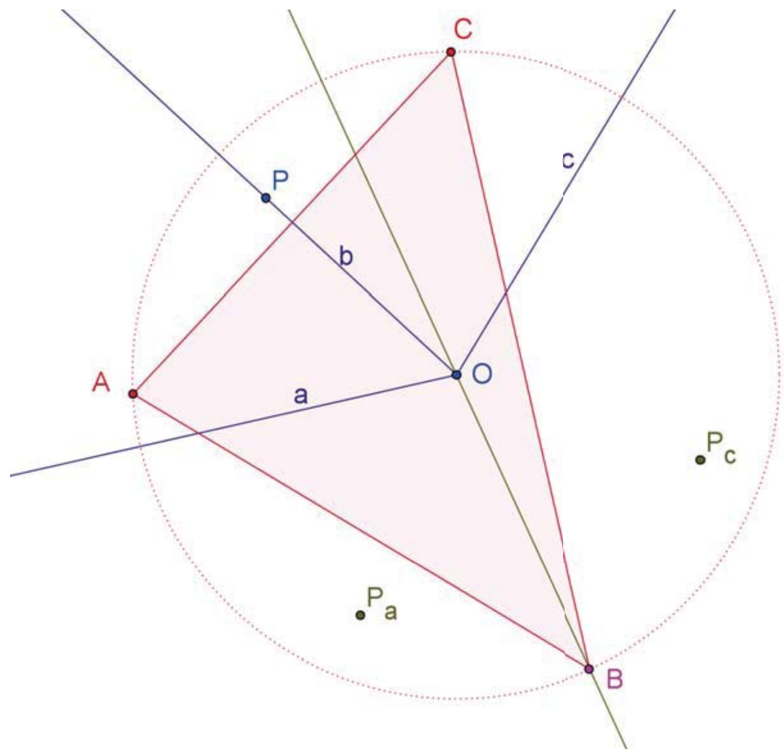

Figure 8.16: Euclidean construction of a triangle with three given side bisectors.

Farkas Bolyai to his son János:

> You must not attempt this approach to parallels. I know this way to its very end. I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave the science of parallels alone ....
>
> I thought I would sacrifice myself for the sake of the truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified to mankind. I accomplished monstrous, enormous labors; my creations are far better than those of others and yet I have not achieved complete satisfaction. ...I turned back when I saw that no man can reach the bottom of the night. I turned back unconsoled, pitying myself and all mankind.
>
> I admit that I expect little from the deviation of your lines. It seems to me that I have been in this regions; that I have traveled past all reefs of this infernal Dead Sea and have always come back with broken mast and torn sail. The ruin of my disposition and my fall date back to this time. I thoughtlessly risked my life and happiness—*aut Caesar aut nihil.*

But the young Bolyai was not deterred by his father's warnings. We exemplify now a second important case occurring in hyperbolic geometry.

Proposition 8.13 (The three reflections theorem for a triangle with vertices on an equidistance line). *Similar to Proposition 9.3 from the section on neutral triangle geometry, we assume that the three side bisectors have a common perpendicular* p. *Let* S_A, S_B, S_C *denote the reflections across the lines through the vertices* A, B, C, *respectively, which are perpendicular to* p. *The compositions of these reflections satisfy the same relations* (8.1).

Proof. We consider the same composite mapping $M := S_a \circ S_b \circ S_c \circ S_B$. This mapping takes point B to itself, and the line p as a set to itself. Hence the foot point of the perpendicular q dropped from B onto p, and indeed all points of perpendicular q, are mapped to themselves. The perpendicular $q \perp p$ is the symmetry axis of S_B. Since mapping M preserves the orientation, it cannot exchange the two half plane of the axis. Hence the mapping M is the identity. □

In the hyperbolic plane, it turns out to be especially interesting to deal with the case left open by the last two propositions.

Proposition 8.14 (Hilbert III.4). *In a hyperbolic plane is given a triangle* $\triangle ABC$, *which has two side bisectors* a *and* c *with a common end* ∞, *and vertex* B *lying in the strip between the two limiting parallel rays. Then all three side bisectors have a common end.*

Proof. The common end can only occur for the two rays on a and c running from the midpoints of the respective side through the interior of the triangle $\triangle ABC$. As a consequence of Pasch's axiom, these rays intersect the third side CA at points P and Q, respectively. Each point of a side bisector has equal distance to the two endpoints of this side. We get together with the triangle inequality

$$|PC| = |PB| < |PQ| + |QB| = |PQ| + |QA| = |PA|$$
$$|QA| = |QB| < |QP| + |PB| = |QP| + |PC| = |QC|$$

Hence the midpoint M of triangle side AC lies between points P and Q, and therefore in the strip between the other two side bisectors a and c. As stated in Proposition 9.2, either all three side bisectors intersect at one point, or neither any two of them intersect. We conclude that the side bisector b cannot intersect neither a or c, since it is assumed these two side bisectors do not intersect. Hence the entire line b lies in the strip between bisectors a and c.

Any ray with vertex M different from $\overrightarrow{M\infty}$ lying on the same side of CA as ∞ does intersect either ray $\overrightarrow{P\infty}$ or ray $\overrightarrow{Q\infty}$—otherwise the latter two rays would not be limiting parallels. Hence all three rays $\overrightarrow{M\infty}$, $\overrightarrow{P\infty}$ and $\overrightarrow{Q\infty}$ are limiting parallels. □

Question. Convince yourself that none of the three opposite rays can be limiting parallels.

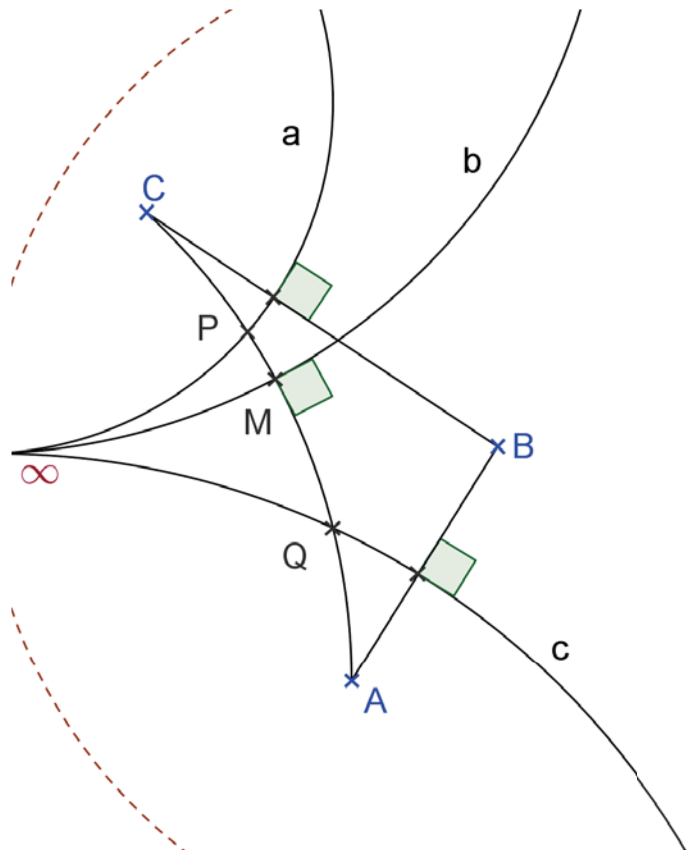

Figure 8.17: Three limiting parallel side bisectors.

Proposition 8.15. *For any triangle in the hyperbolic plane exactly one of the three alternatives occurs:*

(a) *The bisectors of all three sides intersect in one point. The triangle has a circumcircle.*

(b) *The bisectors of all three sides have a common perpendicular. The three vertices lie on an equidistance line.*

(c) *The bisectors of all three sides have a common end.*

Proof. Assume that neither case (a) or (b) occurs. By Proposition 9.2 and Proposition 9.3 from the section about neutral triangle geometry and Hilbert's Proposition 8.8, we conclude that any two of the three side bisectors have a common end.

We may assume, by choosing the names, the rays $\overrightarrow{M_a\infty}$ and $\overrightarrow{M_b\infty}$ from the midpoints of the sides BC and BA, respectively, have the common end ∞. The common end can only occur for the two rays on a and c running from the midpoints of the respective side through the interior of the triangle $\triangle ABC$. The vertex B can only lie in the strip between the two limiting parallel rays. As explained in Proposition 8.14, the third side bisector b contains the ray $\overrightarrow{M_b\infty}$ with the same common end. □

Proposition 8.16. *In a hyperbolic plane is given a triangle $\triangle ABC$, for which all three side bisectors have a common end ∞. Let S_A, S_B, S_C denote the reflections across the lines $\infty A, \infty B, \infty C$, respectively. The compositions of these reflections satisfy the same relations* (8.1).

Conversely, we can now solve the construction problem 8.5 for any triangle in a hyperbolic plane.

Proposition 8.17 (Generalization of Hilbert III.5)**.** *We expect three cases:* [68]

In any Hilbert plane, for any three rays with a common intersection point, there exists a triangle for which these are the side bisectors.

In any Hilbert plane, for any three rays with a common perpendicular, there exists a triangle for which these are the side bisectors.

In a hyperbolic plane, for any three rays with a common end, there exists a triangle for which these are the side bisectors.

In all three cases, the reflections S_a, S_b, S_b across the side bisectors, and appropriate reflections S_A, S_B, S_C across axes through the vertices A, B, C satisfy the relations (8.1).

If the axes of S_a, S_b, S_c have an intersection point O, the axes of reflection S_A, S_B, S_C pass through point O, too.

If the axes of S_a, S_b, S_c have a common perpendicular p, the axes of reflection S_A, S_B, S_C are perpendicular to p, too.

If the axes of S_a, S_b, S_c have a common end ∞, the axes of reflection S_A, S_B, S_C have the end ∞, too.

Proof. We proceed as in the solution of Problem 8.5, which yields the first statement and construct the axis S_B using relation $S_B = s_a \circ S_b \circ S_c$ from (8.1).

Take any point P on the side bisector b and construct its reflection images P_a and P_c across lines a and c, respectively. The perpendicular bisector of P_aP_c is the axis of the reflection S_B. We can choose any point B on this line as one of the vertices of the triangle, and get the other vertices A and C by reflections across the given side bisectors c and a, respectively. In all cases, the axis of the reflection S_B passes through vertex B. The meaning of the axis of the reflection S_B differs in the three cases, in the way as explained. □

[68] In a semi-elliptic plane, or a semi-hyperbolic plane that is not hyperbolic, there are further awkward cases possible.

Proposition 8.18. *For any triangle, the following three statements are equivalent:*

(a) *The perpendicular bisectors of two sides have a common end.*

(b) *The three vertices lie on an horocycle.*

(c) *The bisectors of all three sides have a common end.*

8.6 Construction of Hilbert's field of ends

In any hyperbolic plane, our results allow us to extend the domain of the mapping of reflection across any line and include all ends (which are also called ideal or improper points). We agree that proper segments are never congruent to rays or lines, and proper angles are never congruent to improper angles with an end as vertex. We can extend the obvious and well-known facts about preservation of incidence and congruence by the mapping of reflection by allowing to replace points by ends.

Question. Write down as many different instances of this idea as you find.

Given is any hyperbolic plane. Three different ends are chosen arbitrarily and named $0, 1$ and ∞.

Question. Why do there exist three different ends?

We drop the perpendicular from end 1 onto the line $0\,\infty$ and call the foot point i. This foot point is a point of the hyperbolic plane, but not an end. [69]

Question. Explain the construction to drop the perpendicular from a given end onto a given line.

Remark. As an alternative, we may as well begin by choosing only two ends, and name them 0 and ∞. Then we choose a point on the line $0\,\infty$, naming it i. At this point, we erect the perpendicular, and call one end of it 1.

We construct a field **F**, the elements of which are the ends different from ∞. We now use reflections—and their composition as mappings— to define the operations of addition and multiplication of ends.

8.6.1 Addition of ends

For any end $\alpha \neq \infty$, let S_α denote the reflection across the line $\alpha\,\infty$. The sum of two ends $\alpha, \beta \neq \infty$ is defined by the requirement

$$S_{\alpha+\beta} = S_\alpha \circ S_0 \circ S_\beta \tag{8.2}$$

Because of Proposition 8.17 (Hilbert's proposition III.5), there exists an end γ such that S_γ equals the left-hand side of formula (8.2).

[69] I use names in agreement with the Poincaré half plane model from the previous section.

Question. Give the explicit construction of the sum $\alpha + \beta$.

Answer. Take any point P on the line $0\,\infty$ and construct its reflection images P_α and P_β across lines $\alpha\,\infty$ and $\beta\,\infty$, respectively. The line $(\alpha+\beta)\,\infty$ is the perpendicular bisector of the segment $P_\alpha P_\beta$.

Question. Check that the operation of addition is commutative and associative.

We easily see that 0 is indeed the neutral element of addition. Given is any end $\alpha \neq \infty$. The additive inverse $(-\alpha)$ can be obtained from

$$S_{-\alpha} = S_0 \circ S_\alpha \circ S_0 \tag{8.3}$$

Question. Give the explicit construction of the additive inverse $-\alpha$.

Answer. The line $(-\alpha)\,\infty$ is the mirror image of the line $\alpha\,\infty$ across the symmetry axis $0\,\infty$. Hence we can take any point P on the line $\alpha\,\infty$ and construct its reflection image across line $0\,\infty$. We get a point P' of the line $(-\alpha)\,\infty$, and hence its end $-\alpha$.

The ends in the same half plane of line $0\,\infty$ as 1 are called *positive*, the ends in the opposite half plane are *negative*. I call the hyperbolic half plane containing rays of end 1 the positive semi-region. [70]

Question. Check that the sum of two positive ends is positive.

8.6.2 Multiplication of ends

For any positive end $\mu \neq \infty$, let M_μ denote the reflection across the line $(-\mu)\,\mu$.

Lemma 8.9. *The lines $0\,\infty$ and $(-\mu)\,\mu$ are perpendicular. Hence*

$$S_0 \circ M_\mu = M_\mu \circ S_0 \tag{8.4}$$

commutes for all $\mu > 0$.

Proof. Let p be any point of line $\mu\,\infty$ and P' its mirror image across line $0\,\infty$. Let F be the midpoint of PP'. This is the point were the line $0\,\infty$ is intersected perpendicularly. We draw the rays $\overrightarrow{F\mu}$ and $\overrightarrow{F\,(-\mu)}$. Let G the intersection point of the enclosing line $(-\mu)\,\mu$ with line $0\,\infty$. By ASAL and SAL congruence, we get two pairs of congruent limiting triangles

$$\triangle \mu FP \cong \triangle(-\mu)FP \quad \text{and} \quad \triangle \mu FG \cong \triangle(-\mu)FG$$

The congruent supplementary angles $\angle \mu GF \cong \angle(-\mu)GF$ at vertex G are right angles. □

[70] In the Poincaré half-plane model, this is the quadrant of $u + iv$ with $u > 0$ and $v > 0$.

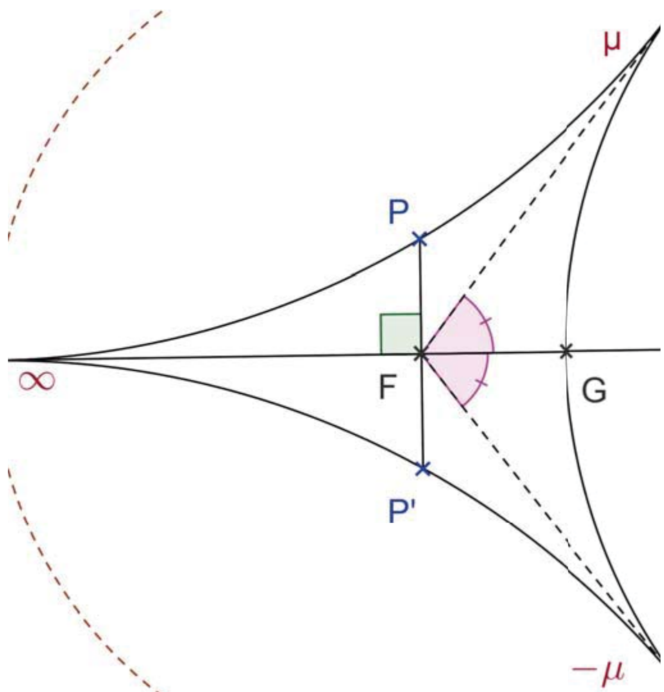

Figure 8.18: The lines $0\,\infty$ and $(-\mu)\,\mu$ are perpendicular.

The product of two positive ends $\beta, \gamma > 0$ is defined by the requirement

$$M_{\beta\gamma} = M_\beta \circ M_1 \circ M_\gamma \tag{8.5}$$

Because of the second part of Proposition 8.17, there exists an end μ such that M_μ equals the left-hand side of formula (8.5).

Question. Give the explicit construction of the product $\beta\gamma$.

Answer. Take any point Q on the line $(-1)\,1$, for example i. Construct its reflection images Q_β and Q_γ across lines $(-\beta)\,\beta$ and $(-\gamma)\,\gamma$, respectively. The line $(-\beta\gamma)\,(\beta\gamma)$ is the perpendicular bisector of the segment $Q_\beta Q\gamma$. Its end $\beta\gamma > 0$ lies in the positive semi-region.

Question. Check that the operation of multiplication is commutative and associative.

We easily see that 1 is indeed the neutral element of multiplication. Given is any positive end $\alpha > 0$. The multiplicative inverse α^{-1} can be obtained from

$$M_{\alpha^{-1}} = M_1 \circ M_\alpha \circ M_1 \tag{8.6}$$

Question. Give the explicit construction of the multiplicative inverse α^{-1}.

Answer. The line $(-\alpha^{-1})\,\alpha^{-1}$ is the mirror image of the line $(-\alpha)\,\alpha$ across the symmetry axis $(-1)\,1$. Hence we can take any point Q on the line $(-\alpha)\,\alpha$ and construct its reflection image across line $(-1)\,1$. We get a point Q' of the line $(-\alpha^{-1})\,\alpha^{-1}$. We drop the perpendicular onto line $0\,\infty$. Because of Lemma 8.9, this is the required line $(-\alpha^{-1})\,\alpha^{-1}$.

8.6.3 The distributive law

Lemma 8.10.

$$M_\gamma \circ M_\alpha = M_{\gamma/\alpha} \circ M_1 \tag{8.7}$$
$$M_\gamma \circ M_\alpha \circ M_\gamma = M_{\gamma^2\alpha^{-1}} \tag{8.8}$$
$$M_\gamma(\alpha) = \gamma^2\alpha^{-1} \tag{8.9}$$

The ends α and $\gamma^2\alpha^{-1}$ are mirror images across the line $(-\gamma)\,\gamma$.

Proof. The first formula (8.7) can be checked using the definition 8.5 of the product and the fact that M_α is an involution. To check the second formula (8.8) is left to the reader, too.

Let $\beta = M_\gamma(\alpha)$ be the mirror image of end α across the line $(-\gamma)\,\gamma$. The mapping $M_\gamma \circ M_\alpha \circ M_\gamma$ on the left-hand side of the second formula (8.8) takes $\beta \mapsto \alpha \mapsto \alpha \mapsto \beta$ and exchanges the ends 0 and ∞. Hence it is the reflection across the perpendicular dropped form β onto line $0\,\infty$, and hence

$$M_\gamma \circ M_\alpha \circ M_\gamma = M_\beta$$

Now the second formula (8.8) implies that $\beta = \gamma^2\alpha^{-1}$, and hence formula (8.9). In other words, the mirror image of end α across the line $(-\gamma)\,\gamma$ is $\gamma^2\alpha^{-1}$. □

Problem 8.6. *Prove by a similar argument that*

$$S_\gamma(\alpha) = 2\gamma - \alpha \tag{8.10}$$

Lemma 8.11.

$$S_{\gamma^2\alpha^{-1}} = M_\gamma \circ M_1 \circ S_{\alpha^{-1}} \circ M_1 \circ M_\gamma \tag{8.11}$$
$$S_{\gamma^2\alpha} = M_\gamma \circ M_1 \circ S_\alpha \circ M_1 \circ M_\gamma \tag{8.12}$$

Proof. The right-hand side of formula (8.11) maps

$$\infty \mapsto \infty \quad \text{and} \quad \beta \mapsto \alpha \mapsto \alpha^{-1} \mapsto \alpha^{-1} \mapsto \alpha \mapsto \beta$$

Hence this mapping equals the reflection S_β, again with $\beta = \gamma^2\alpha^{-1}$. It can be left to the reader to check the last formula (8.12). □

Lemma 8.12.

$$\gamma^2\alpha + \gamma^2\beta = \gamma^2(\alpha + \beta)$$

Proof. Use the definition 8.2 of the sum on the left-hand side, formula (8.12) twice, and the commutativity (8.4):

$$\begin{aligned} S_{\gamma^2\alpha+\gamma^2\beta} &= S_{\gamma^2\alpha} \circ S_0 \circ S_{\gamma^2\beta} \\ &= M_\gamma \circ M_1 \circ S_\alpha \circ M_1 \circ M_\gamma \circ S_0 \circ M_\gamma \circ M_1 \circ S_\beta \circ M_1 \circ M_\gamma \\ &= M_\gamma \circ M_1 \circ S_\alpha \circ S_0 \circ M_1 \circ M_\gamma \circ M_\gamma \circ M_1 \circ S_\beta \circ M_1 \circ M_\gamma \\ &= M_\gamma \circ M_1 \circ S_\alpha \circ S_0 \circ S_\beta \circ M_1 \circ M_\gamma \\ &= M_\gamma \circ M_1 \circ S_{\alpha+\beta} \circ M_1 \circ M_\gamma = S_{\gamma^2(\alpha+\beta)} \end{aligned}$$

□

Lemma 8.13. *For every $\mu > 0$, there exists a square root such that $\gamma^2 = \mu$ and $\gamma > 0$.*

Proof. Let i and K be the intersection points of the lines $(-1)\,1$ and $(-\mu)\,\mu$ intersect line $0\,\infty$ at right angle, respectively. We erect the perpendicular on the midpoint C of segment $i\,K$. Let $-\gamma$ and γ be the ends of the perpendicular. We easily check that this construction implies

$$M_\mu = M_\gamma \circ M_1 \circ M_\gamma$$

Indeed, both sides map point K to itself, the right-hand side via $K \mapsto i \mapsto i \mapsto K$. Both side leave the line $0\,\infty$ invariant and reverse the orientation. Hence these mappings are equal. By definition of the field operations this means that $\mu = \gamma^2$. Thus we have constructed the required square root. □

Problem 8.7. *For every α, there exists a half such that $2\eta = \alpha$. Use*

$$S_\alpha = S_\eta \circ S_0 \circ S_\eta$$

and get this fact by an analogous argument.

Proposition 8.19. *Hilbert's field* $\mathbf{F}$ *of ends is an ordered Euclidean field.*

Proof. We have checked that the sum of two positive ends is positive. Our definition makes the product of two positive ends positive, too. One defines a total order by requiring that $\alpha < \beta$ if and only if $\beta - \alpha > 0$. Thus one gets an ordered field. To check whether $\mathbf{F}$ is Euclidean, we need to find a square root for any given $\mu > 0$, as explained in Lemma 8.13 above. The distributive law is now easily deducted from Lemma 8.12. □

8.7 Reconstruction of the complex field and the half-plane model

From Hilbert's field of ends $\mathbf{F}$, we can construct the half-plane $H = \{(u, v) : u, v \in \mathbf{F} \text{ and } v > 0\}$. We endow the plane with the Euclidean geometry and get a Euclidean plane, precisely with the properties stated by Definition 7.3 in the section about the natural axiomatization of geometry. In a second step, we use this Euclidean geometry as the underlying theory, upon which to reconstruct the Poincaré's half-plane model of hyperbolic geometry.

Proposition 8.20. *Given are ends $u, v \in \mathbf{F}$ with $v > 0$. All lines with ends u_1, u_2 such that*

$$u_1 u_2 - u\,(u_1 + u_2) + u^2 + v^2 = 0$$

go through one point P. Especially, this point is the intersection of the lines $u\,\infty$ and $-\sqrt{u^2+v^2}\,\sqrt{u^2+v^2}$.

Proof. All lines through point i, except $0\,\infty$, have ends $u_1, u_2 \in \mathbf{F}$ satisfying

$$u_1\,u_2 + 1 = 0$$

Problem 8.8. *Convince yourself of this statement.*

Equations (8.10) and (8.9) imply

$$u + \beta = S_{u/2} \circ S_0(\beta) \tag{8.13}$$

$$v\alpha = M_{\sqrt{v}} \circ M_1(\alpha) \tag{8.14}$$

for any ends u, v, α and β.

The mapping $M_{\sqrt{v}} \circ M_1$ maps ends $u_1 \mapsto vu_1 =: u_1'$ and $u_2 \mapsto vu_2 =: u_2'$ and point $i \mapsto Q$, hence

$$Q = M_{\sqrt{v}} \circ M_1(i)$$

Hence all lines through point Q, except $0\,\infty$, have ends u_1', u_2' satisfying

$$u_1' u_2' + v^2 = 0$$

The mapping $S_{u/2} \circ S_0$ takes ends $u_1' \mapsto u + u_1' = u + vu_1 =: u_1''$ and $u_2' \mapsto u + u_2' = u + vu_2 =: u_2''$ and point $Q \mapsto P$, hence

$$P = S_{u/2} \circ S_0(Q)$$

Hence all lines through point P, except $u\,\infty$, have ends u_1'', u_2'' satisfying

$$u_1'' u_2'' - u\,(u_1'' + u_2'') + u^2 + v^2 = 0$$

□

Point P is obtained from the point i by the mapping $M_{\sqrt{v}} \circ M_1$, which is called a *translation along* $0\,\infty$, The mapping $S_{u/2} \circ S_0$, used in a second step, is called a *rotation around* ∞.

Every point P of the hyperbolic plane can be obtained in this way. Indeed, given any point P, there exists a ray $P\,\infty$. Let $u \in \mathbf{F}$ be the second end of line $P\,\infty$. Let Q the mirror image of P across the line $(u/2)\,\infty$. Let $-\gamma\,\gamma$ be the perpendicular bisector of segment $i\,Q$, and finally put $v := \gamma^2$. According to this construction, we have obtained ends $u, v \in \mathbf{F}$, $v > 0$ and got mappings such that

$$\begin{aligned} P &= S_{u/2} \circ S_0(Q) \\ Q &= M_{\sqrt{v}} \circ M_1(i) \\ P &= S_{u/2} \circ S_0 \circ M_{\sqrt{v}} \circ M_1(i) \end{aligned}$$

We give the point P the coordinates $u + iv$. We see by Proposition 8.20 that the Euclidean equation of a hyperbolic line through point P with ends u_1 and u_2 is

$$\left(u - \frac{u_1'' + u_2''}{2}\right)^2 + v^2 = \left(\frac{u_1'' - u_2''}{2}\right)^2$$

except the additional case of "vertical" line with arbitrary v and u as given by point P. In the interpretation of the Euclidean coordinate geometry, we see that the hyperbolic lines through an arbitrary point P are depicted as circular arcs and lines perpendicular to the u-axis. The u-axis becomes the line at infinity for the half-plane model.

The construction above shows that enough rigid motions exist in order map any point P and ray with vertex P to the point i and the ray $\overrightarrow{i\infty}$. Hence one can define hyperbolic congruence of segments and angles. In a second step, we use the Euclidean geometry in the plane $\{(u, v) : u, v \in \mathbf{F}\}$ as the underlying theory to reconstruct Poincaré's half-plane model of hyperbolic geometry.

Theorem 8.2 (Reconstruction of the half-plane model). *Any hyperbolic plane is isomorphic to the Poincaré model with the half-plane*

$$H = \{(u, v) : u, v \in \mathbf{F} \ \ \textit{and} \ \ v > 0\}$$

over the Hilbert field $\mathbf{F}$ *of ends.*

8.8 The angle unboundedness postulate

Proposition 8.21. *In a hyperbolic Hilbert plane, Aristole's Angle Unboundedness Axiom 10.2 holds.*

Proof. Let the acute angle $\theta = \angle(m, n)$ be given. Reflection across the side n yields the double angle $2\theta = \angle(m, m')$. As shown in the proof of Hilbert's foundation of hyperbolic geometry, the two rays m and m' have an inclosing line l. One checks that l and n intersect perpendicularly, say at point P.

Let p be the ray on l with vertex P which is asymptotically parallel to ray m. Let μ denote their common end. We transfer any given segment onto this ray to produce the (arbitrarily long) segment PQ. We drop the perpendicular from Q onto ray m. Since we are staying on one side of n, we obtain the foot point F. The angle $\angle FQ\mu$ is an acute angle of parallelism.

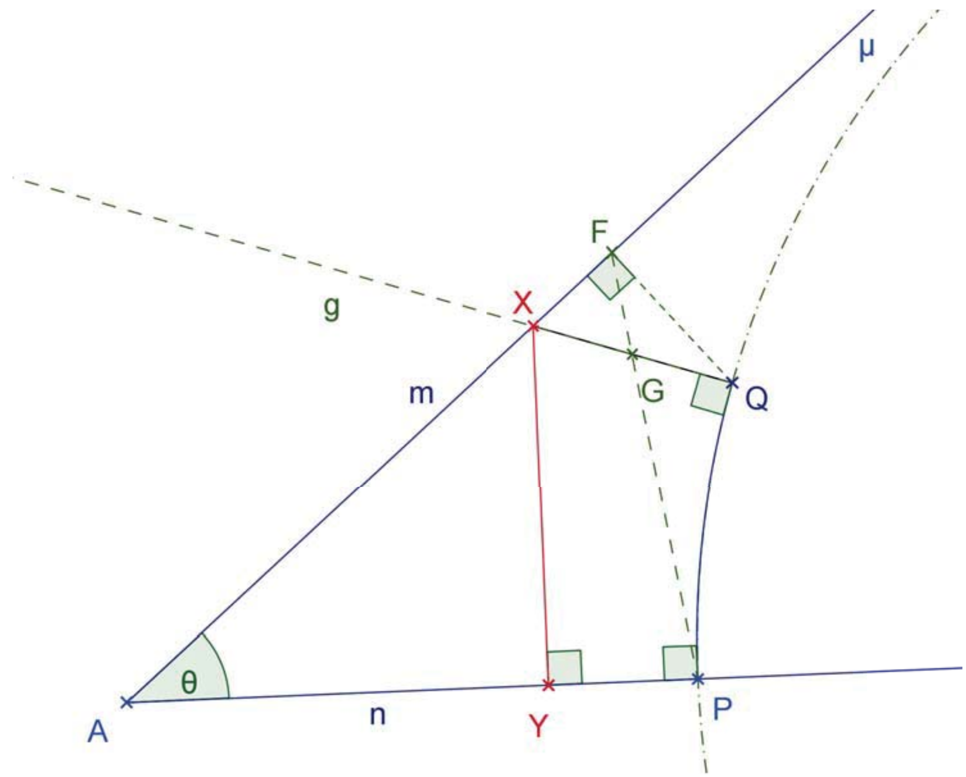

Figure 8.19: Existence of asymptotically parallel rays implies Aristole's Axiom.

Hence the angle $\angle PQF$ is obtuse. We erect onto line l the perpendicular ray g at Q inside this angle. This ray is inside the angle $\angle PQF$, and by the crossbar theorem, intersects segment PF, say at point G.

We apply Pasch's axiom to triangle $\triangle APF$ and line g, where A is the vertex of angle θ. Since g and n are both perpendicular to PQ, they are parallel. But ray g and side PF intersect in point G. Hence by Pasch's axiom, line g intersects the third side AF. Call X the intersection point.

We drop the perpendicular from point H onto line n and obtain the foot point Y. Finally, we have the Lambert quadrilateral $\square QXYP$, with an acute angle $\chi = \angle QXY$ at vertex X. By Lemma 10.1 either side adjacent to the acute angle is longer than the respectively opposite side. Hence we conclude

$$XY > PQ$$

as required for Aristole's angle unboundedness axiom to hold. $\square$

Proposition 8.22. *Given is a semi-hyperbolic Hilbert plane, for which Aristole's Axiom holds.*

It two lines have a common perpendicular, then the distance between two lines is arbitrarily large. In other words, for any given segment, there exists a point on one of the two lines from which the distance to the other line is longer than the given segment.

Corollary 65. *In a hyperbolic plane, the distance between two lines with a common perpendicular is arbitrarily large.*

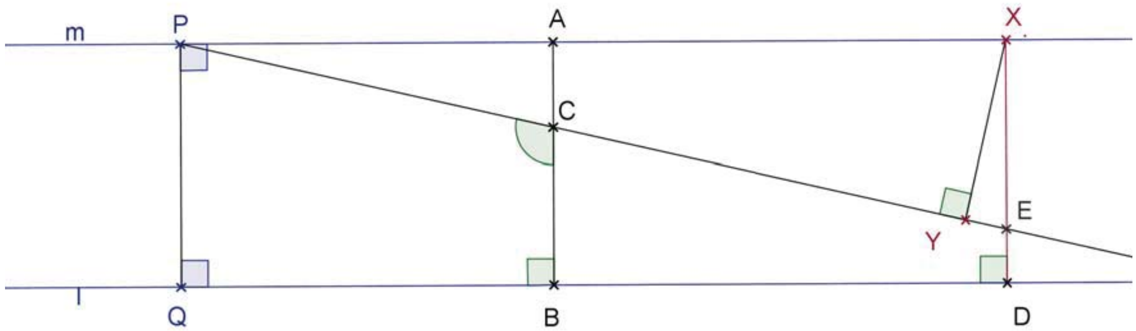

Figure 8.20: In a hyperbolic plane, the distance between two lines with a common perpendicular is arbitrarily large.

Proof. Let PQ the segment on the common perpendicular from line m onto line l. From any second point A of line m, we drop the perpendicular onto line l and get the foot point B. Now $\square QBAP$ is a Lambert quadrilateral, with an acute angle $\angle PAB$.

We drop the perpendicular from point P onto line AB and get the foot point C. We have produced a second Lambert quadrilateral $\square QBCP$, which has an acute angle $\angle QPC$. The ray $\overrightarrow{PC}$ lies inside the right angle $\angle QPA$, and hence point C lies between A and B.

We use Aristole's axiom for the angle $\theta = \angle APC$. Hence there exists a point X on the side $\overrightarrow{PA}$ for which the segment XY dropped onto the other side is longer than the given segment ST:

$$XY > ST$$

We drop from point X the perpendicular onto line l and get the foot point D. The points X and D lie on different sides of line PC, since points A and X lie on one side, and the points Q, B and D lie on the other side. Hence the segment XD intersects line PC in a point E, and

$$XD > XE$$

The segment XE is the hypothenuse of the right triangle $\triangle XYE$, and hence its longest side

$$XE > XY$$

Hence for any given segment ST, there exists a point X on line m such that the distance

$$XD > XE > XY > ST$$

from point X the foot point D on line l is longer than the given segment ST. $\square$