9 Gauss' Differential Geometry and the Pseudo-Sphere

9.1 Introduction

Through the work of Gauss on differential geometry, it became clear—after a painfully slow historic process—that there is a model of hyperbolic geometry on surfaces of constant negative Gaussian curvature. One particularly simple such surface is the *pseudo-sphere*.

According to Morris Kline, it is not clear whether Gauss himself already saw this non-Euclidean interpretation of his geometry of surfaces. Continuing Gauss work, Riemann and Minding have thought about surfaces of constant negative curvature. Neither Riemann nor Minding did relate curved surfaces to hyperbolic geometry (Morris Kline III, p.888 etc). But, independently of Riemann, Beltrami finally recognized that surfaces of constant curvature are non-Euclidean spaces. Due to the ideas forwarded by Gauss, mathematicians have in the end advanced to the concept of a curved surface as a space of its own interest. Gauss' work implies that there are non-Euclidean geometries on surfaces regarded as spaces in themselves. An obvious and important idea is finally spelled out!

As we explain in detail below, Beltrami shows that one can realize a *piece* of the hyperbolic plane on a rotation surface of negative constant curvature. This surface is called a *pseudo-sphere*. But this new discovery comes with a disappointment: by a result of Hilbert, there is no regular analytic surface of constant negative curvature on which the geometry of the *entire* hyperbolic plane is valid (see Hilbert's Foundations of Geometry, appendix V).

Concerning models of hyperbolic geometry, the final outcome turns out to be a trade off between the pseudo-sphere and the Poincaré disk. Both have their strengths and weaknesses. The pseudo-sphere is a model for a *limited portion* of the hyperbolic plane. Both angle and length are represented correctly. The arc length of a geodesic is the *correct hyperbolic distance*. Furthermore, because of the constant Gaussian curvature, on the pseudo-sphere a figure may be shifted about and just bending will make it conform to the surface. The situation is similar to the more familiar case of Euclidean geometry on a circular cylinder or cone. As everybody knows, on a circular cylinder, a plane figure can be fitted by simply bending it, without stretching and shrinking.

On the other hand, only the Poincaré disk is a model for the *entire* hyperbolic plane. Here only angles are still represented correctly, but the price one finally has to pay is that *hyperbolic distances are distorted*. The hyperbolic lines become circular arcs, perpendicular to the ideal boundary. One can see the distortion easily in Esher's superb artwork, based on tiling of the hyperbolic plane with congruent figures.

The trade off just explained makes the isometry between the pseudo-sphere into the Poincaré disk especially interesting. One such isometric mapping is explicitly constructed below. Hilbert's result gets rather natural, too. As explained below, in the sense of hyperbolic geometry, the boundary of the pseudo-sphere turns out to be an arc of a horocyle.

9.2 About Gauss' differential geometry

Karl Friedrich Gauss had devoted an immense amount of work to geodesy and map making, starting 1816. This stimulus leads to his definitive paper in differential geometry of 1827: "Disquisitiones Generales circa Surperficies Curvas" . In this work, Gauss introduces the basics of curved surfaces, and goes far beyond. The real benefit is that, due to the ideas forwarded by Gauss, mathematicians have in the end advanced to the concept of a curved surface as a space of its own interest.

To begin with, one imagines a curved surface to be embedded into three dimensional space $\mathbf{R}^3$, and given by some parametric equations

$$x = x(u, v)\,,\ y = y(u, v)\,,\ z = z(u, v) \tag{9.1}$$

The distance ds of neighboring points on the surface with parameters (u, v) and $(u + du, v + dv)$ is given by the first fundamental form

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \tag{9.2}$$

The first fundamental form is straightforward to calculate from the parametric equations (9.1) since

$$\begin{aligned} E &= x_u^2 + y_u^2 + z_u^2 \\ F &= x_u x_v + y_u y_v + z_u z_v \\ G &= x_v^2 + y_v^2 + z_v^2 \end{aligned} \tag{9.3}$$

follows from elementary vector calculus.

The <u>geodesics</u> on curved surfaces are defined to be the shortest curves lying on the given surface, connecting any two given points. Gauss' work sets up the differential equation for the geodesics. Gauss introduces the two <u>main curvatures</u>, called κ_1, κ_2. They turn out to be simply the extremal curvatures of normal sections of the surface. A new important feature is the <u>Gaussian curvature</u>, called K. Gauss shows that $K = \frac{LN-M^2}{EG-F^2}$, the quotient of the determinants of the second and first fundamental form. But, even simpler, the Gaussian curvature turns out to be the product of the two principle curvatures:

$$K = \kappa_1 \kappa_2 \tag{9.4}$$

Gauss shows the remarkable fact that this curvature is preserved during the process of bending the curved surface inside a higher dimensional space, without stretching, contracting or tearing it. On the contrary, the two main curvatures are changed by flexing the surface.

There are actually at least two different proofs for this fact contained in Gauss' work. The first one depends on Gauss' characteristic equation

$$K = \frac{1}{2H}\frac{\partial}{\partial u}\left[\frac{F}{EH}\frac{\partial E}{\partial v} - \frac{1}{H}\frac{\partial G}{\partial u}\right] + \frac{1}{2H}\frac{\partial}{\partial v}\left[\frac{2}{H}\frac{\partial F}{\partial u} - \frac{1}{H}\frac{\partial E}{\partial v} - \frac{F}{EH}\frac{\partial E}{\partial u}\right] \tag{9.5}$$

where $H = \sqrt{EG - F^2}$. Obviously, any such equation implies that the Gaussian curvature depends only on the first fundamental form. The first fundamental form is preserved, if one bends the curved surface in three space, without stretching, contracting or tearing it. Therefore the functions E, F, G, H which determine the first fundamental form depend only on the parameters (u, v), but do not depend at all on how—or even whether at all—the surface lies in a three dimensional space. Because of the Gauss' characteristic equation (9.5), the same is true for the Gaussian curvature K. Because of all that, one says that the Gaussian curvature is an intrinsic property of the curved surface.

9.3 Riemann metric of the Poincaré disk

Proposition 9.1 (Riemann Metric for Poincaré's Model). *In the Poincaré model, the infinitesimal hyperbolic distance ds of points with coordinates (x, y) and $(x+dx, y+dy)$ is*

$$(ds_D)^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} \tag{9.6}$$

Reason. The fact that angles are measured in the usual Euclidean way implies that $ds^2 = E(x, y)(dx^2 + dy^2)$. The rotational symmetry around the center O implies that $E(x, y) = E(\sqrt{x^2 + y^2}, 0)$. Hence

$$ds^2 = E(\sqrt{x^2 + y^2}, 0)(dx^2 + dy^2) \tag{9.7}$$

Now it is enough to calculate the distance of the points $(x, 0)$ and $(x + dx, 0)$. The hyperbolic distance of a point $(x, 0)$ from the center $(0, 0)$ is $2\tanh^{-1} x$, as we have derived in Proposition 1.2 in the section on the Poincaré disk model. See formula (1.11) there, which is of course the primary origin of the hyperbolic distance! Taking the derivative by the variable x yields

$$\begin{aligned} \frac{ds}{dx} &= \frac{d}{dx}(2\tanh^{-1} x) = \frac{d}{dx}\ln\frac{1+x}{1-x} = \frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2} \\ ds^2 &= \frac{4}{(1-x^2)^2}\,dx^2 \end{aligned} \tag{9.8}$$

Hence $E(x, 0) = \frac{4}{(1-x^2)^2}$ and

$$E(\sqrt{x^2 + y^2}, 0) = \frac{4}{(1 - x^2 - y^2)^2} \tag{9.9}$$

Now formulas (9.7) and (9.9) imply the claim (9.14). □

Problem 9.1 (Hyperbolic circumference of a circle). *Calculate the circumference of a circle of hyperbolic radius R. We use the Poincaré disk, put the center of the circle at the center O of the disk. In polar coordinates, the Riemann metric is*

$$ds^2 = 4\,\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2} = 4\,\frac{dr^2 + r^2 d\theta^2}{(1 - r^2)^2}$$

(a) *Calculate the hyperbolic length R of a segment OA with Euclidean length $|OA| = r < 1$.*

(b) *Get the circumference $C = \int ds$ of the circle around O, at first in terms of the Euclidean radius $|OA| = r < 1$.*

(c) *Get the circumference C of this circle in terms of the hyperbolic radius R.*

Solution. We take O as center of the circle, and point A on the circumference. Let $r = |OA|$ denote the Euclidian radius, and $R = s(O, A)$ be the hyperbolic radius.

The hyperbolic radius R can be found directly for the Riemann metric (9.14). One needs partial fractions to do the integral.

$$R = \int_0^r ds = \int_0^r \frac{2dr}{1 - r^2} = \int_0^r \left[\frac{1}{1-r} + \frac{1}{1+r}\right] dr = [-\ln(1-r) + \ln(1+r)]_0^r$$
$$= 2\tanh^{-1} r$$

Remark. Of course, we can go back once more to Proposition 1.2, formula (1.11) from the section on the Poincaré disk model and get $R = s(O, A) = 2\tanh^{-1}|OA| = 2\tanh^{-1} r$.

We solve $R = 2\tanh^{-1} r$ for the Euclidean radius and get

$$r = \tanh\frac{R}{2} = \frac{e^{R/2} - e^{-R/2}}{e^{R/2} + e^{-R/2}} = \frac{e^R - 1}{e^R + 1}$$

For the usual Euclidean polar coordinates (r, θ) we get the Euclidean arc length:

$$L_{Eucl} = \int_0^{2\pi} \sqrt{dx^2 + dy^2} = \int_0^{2\pi} \sqrt{dr^2 + r^2 d\theta^2} = r\int_0^{2\pi} d\theta = 2\pi r \tag{9.10}$$

The first line holds for any smooth curve. In the second line, we go to the special case of a circle . For a circle, the coordinate r is constant and hence $dr = 0$, and the factor r can be pulled out of the integral.

Now the distance along the circumference is measure in the hyperbolic metric (9.14) from Proposition 9.1. Hence the calculation above is modified to

$$\begin{aligned} L_{hyp} &= \int_0^{2\pi} 2\frac{\sqrt{dx^2 + dy^2}}{1 - x^2 - y^2} = \int_0^{2\pi} 2\frac{\sqrt{dr^2 + r^2 d\theta^2}}{1 - r^2} \\ &= \frac{2r}{1 - r^2}\int_0^{2\pi} d\theta = \frac{4\pi r}{1 - r^2} \end{aligned} \tag{9.11}$$

We have found the correct hyperbolic arc length. But still, one needs to use the formula $r = \frac{e^R-1}{e^R+1}$ to express r in terms of the hyperbolic distance R.

$$\begin{aligned} L_{hyp} &= \frac{4\pi r}{1-r^2} = \frac{4\pi(e^R-1)}{[1-(\frac{e^R-1}{e^R+1})^2](e^R+1)} \\ &= \frac{4\pi(e^R-1)(e^R+1)}{(e^R+1)^2-(e^R-1)^2} = \frac{4\pi(e^{2R}-1)}{4e^R} \\ &= \pi(e^R-e^{-R}) = 2\pi\sinh R \end{aligned} \tag{9.12}$$

□

Proposition 9.2 (The circumference of a circle). *In hyperbolic geometry, the circumference of a circle of hyperbolic radius R is $2\pi\sinh R$.*

Problem 9.2. *The hyperbolic circumference of a circle is much larger than the Euclidean circumference. Let $R = 1, 2, 5, 10$ and estimate how many times the radius fits around the circumference of a circle of that radius.*

Answer. A simple calculation yields

R	$(2\pi\sinh R)/R$
1	7.38
2	11.39
5	93.25
10	6919.82

Problem 9.3 (Hyperbolic area of a circle). *For a circle of hyperbolic radius R, calculate the area A. Again, we use the Poincaré disk, put the center of the circle at the center O of the disk. For the area, we use the formula from differential geometry*

$$A = \int_0^{2\pi}\int_0^r \sqrt{EG-F^2}\,dr\,d\theta$$

The first fundamental form is the Riemann metric. It has been already given by formula (9.14), and transformed to polar coordinates in the previous problem.

$$ds^2 = 4\frac{dx^2+dy^2}{(1-x^2-y^2)^2} = 4\frac{dr^2+r^2d\theta^2}{(1-r^2)^2} = Edr^2 + 2Fdrd\theta + Gd\theta^2 \tag{9.13}$$

(a) *Get the area of the circle, at first in terms of the Euclidean radius $|OA| = r < 1$.*

(b) *Get the area A of this circle in terms of the hyperbolic radius R.*

(c) *Check that*

$$\frac{dA}{dR} = C$$

(a) The first fundament form (9.13) yields

$$H = \sqrt{EG - F^2} = \frac{4r}{(1-r^2)^2}$$

and hence the hyperbolic area of a circle of Euclidean radius r is

$$A = \int_0^{2\pi} \int_0^r \sqrt{EG - F^2}\, dr\, d\theta = 2\pi \int_0^r \frac{4r\, dr}{(1-r^2)^2}$$

This integral is solved with the substitution $u = r^2$ and $du = 2rdr$.

$$A = 2\pi \int_0^u \frac{2\, du}{(1-u)^2} = \left[\frac{4\pi}{(1-u)}\right]_0^u = \frac{4\pi}{(1-r^2)} - 4\pi = \frac{4\pi r^2}{(1-r^2)}$$

This is the area in terms of the Euclidean radius $|OA| = r < 1$.

(b) The hyperbolic radius R has already been calculated in the previous problem. We solve $R = 2\tanh^{-1} r$ for the Euclidean radius and get

$$r = \tanh\frac{R}{2} = \frac{e^{R/2} - e^{-R/2}}{e^{R/2} + e^{-R/2}} = \frac{e^R - 1}{e^R + 1}$$

$$r^2 = \frac{(e^R - 1)^2}{(e^R + 1)^2} \quad \text{and} \quad 1 - r^2 = \frac{(e^R + 1)^2 - (e^R - 1)^2}{(e^R + 1)^2} = \frac{4e^R}{(e^R + 1)^2}$$

Now plug this formula into the result from part (a) and get

$$A = \frac{4\pi r^2}{(1-r^2)} = \frac{\pi(e^R - 1)^2}{e^R} = \pi(e^R - 1 + e^{-R}) = 2\pi(\cosh R - 1)$$

An alternative formula is

$$A = \frac{\pi(e^R - 1)^2}{e^R} = \pi(e^{R/2} - e^{-R/2})^2 = 4\pi \sinh^2\frac{R}{2}$$

(c) We have obtained in Proposition 9.2 from the section on the Poincaré disk model, that the hyperbolic circle of radius R has the circumference $C = 2\pi \sinh R$. On the other hand, differentiating the result of (b) gives

$$\frac{dA}{dR} = 2\pi \frac{d \cosh R}{dR} = 2\pi \sinh R = C$$

as to be shown.

Problem 9.4. *Use the fundamental form for the Poincaré disk model*

$$(ds_D)^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} \tag{9.14}$$

to calculate its Gaussian curvature.

Answer. Formula (9.14) implies that the functions in the first fundamental form are $E = G = H = 4(1 - x^2 - y^2)^{-2}$ and $F = 0$. Hence, with $x = u$ and $v = y$, we get from formula (9.5)

$$K = \frac{1}{2E}\frac{\partial}{\partial x}\left[-\frac{1}{E}\frac{\partial E}{\partial x}\right] + \frac{1}{2E}\frac{\partial}{\partial y}\left[-\frac{1}{E}\frac{\partial E}{\partial y}\right] = -\frac{1}{2E}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\ln E$$
$$= +\frac{1}{E}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\ln(1 - x^2 - y^2) = +\frac{1}{E}\left(\frac{\partial}{\partial x}\frac{-2x}{(1 - x^2 - y^2)} + \frac{\partial}{\partial y}\frac{-2y}{(1 - x^2 - y^2)}\right)$$
$$= \frac{-2}{E}\left(\frac{1 - x^2 - y^2 + 2x^2}{(1 - x^2 - y^2)^2} + \frac{1 - x^2 - y^2 + 2y^2}{(1 - x^2 - y^2)^2}\right) = \frac{(-2)\cdot 2}{4} = -1$$

By the way, the result $K = -1$ motivates the annoying factor 4 in formula (9.14).

9.4 Riemann metric of Klein's model

Proposition 9.3 (Hilbert-Klein Metric). *In the Klein model, the infinitesimal hyperbolic distance ds of points with coordinates (X, Y) and $(X + dX, Y + dY)$ is*

$$ds^2 = \frac{dX^2 + dY^2 - (XdY - YdX)^2}{(1 - X^2 - Y^2)^2} \tag{9.15}$$

Proof. We shall derive this metric using the transformation from the Poincaré to the Klein model. As stated in Proposition 3.1, the mapping from a point P in Poincaré's model to a point K in Klein's model is

$$|OK| = \frac{2\,|OP|}{1 + |OP|^2} \tag{3.1}$$

requiring that the rays $\overrightarrow{OP} = \overrightarrow{OK}$ are identical. We use Cartesian coordinates and put $P = (x, y)$ for Poincaré's model and $K = (X, Y)$ for the points in Klein's model. Finally we put $r^2 = x^2 + y^2$ and $R^2 = X^2 + Y^2$. From the mapping (3.1), we get

$$X = \frac{2x}{1 + r^2} \quad \text{and} \quad Y = \frac{2y}{1 + r^2} \tag{9.16}$$

The Riemann metric for Poincaré's model has been derived in Proposition 9.1 to be

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2} = E\,dx^2 + 2F\,dxdy + G\,dy^2$$

Here E, F, G denotes the fundamental form for the Poincaré model in terms of (x, y). In the following we shall use the matrix

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \frac{4}{(1 - x^2 - y^2)^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{9.17}$$

From the fact that the transformation from Poincaré's to Klein's model is a passive coordinate transformation, we know that the infinitesimal hyperbolic distance ds of points is left invariant. Because of the invariance, the fundamental form $\overline{E}, \overline{F}, \overline{G}$ for the Klein model has to satisfy

$$ds^2 = \overline{E}\, dX^2 + 2\overline{F}\, dX dY + \overline{G}\, dY^2 = E\, dx^2 + 2F\, dxdy + G\, dy^2$$

We take for now (x, y) as independent variables. From calculus, we know that by means of the chain rule

$$\begin{bmatrix} \frac{\partial X}{\partial x} & \frac{\partial Y}{\partial x} \\ \frac{\partial X}{\partial y} & \frac{\partial Y}{\partial y} \end{bmatrix} \begin{bmatrix} \overline{E} & \overline{F} \\ \overline{F} & \overline{G} \end{bmatrix} \begin{bmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \tag{9.18}$$

It now remains to carry out the arithmetic. The superscript ^{T} denotes transposition of matrices and the superscript $^{-1}$ denote inversion of matrices. As usual, we use

$$\frac{D\,X}{Dx} = \begin{bmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{bmatrix}$$

as shorthand for the Jacobi matrix of the transformation (9.16). We need to solve the equation (9.18) for the new fundamental form $\overline{E}, \overline{F}, \overline{G}$ to obtain

$$\begin{aligned} \left[\frac{D\,X}{Dx}\right]^T \begin{bmatrix} \overline{E} & \overline{F} \\ \overline{F} & \overline{G} \end{bmatrix} \left[\frac{D\,X}{Dx}\right] &= \begin{bmatrix} E & F \\ F & G \end{bmatrix} \\ \begin{bmatrix} \overline{E} & \overline{F} \\ \overline{F} & \overline{G} \end{bmatrix} &= \left[\frac{D\,X}{Dx}\right]^{T,-1} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \left[\frac{D\,X}{Dx}\right]^{-1} \end{aligned}$$

The Jacobi matrix of the transformation (9.16) is obtained explicitly from equations (9.16) to be

$$\frac{D\,X}{Dx} = \frac{2}{(1+x^2+y^2)^2} \begin{bmatrix} 1 - x^2 + y^2 & -2xy \\ -2xy & 1 + x^2 - y^2 \end{bmatrix}$$

The determinant is

$$\operatorname{Det} \frac{D\,X}{Dx} = \frac{4}{(1+r^2)^4} \left[1 - (x^2 - y^2)^2 - 4x^2y^2\right] = \frac{4}{(1+r^2)^4} \left[1 - r^4\right] = \frac{4(1-r^2)}{(1+r^2)^3}$$

Hence the inverse turned out to be

$$\begin{aligned} \left[\frac{D\,X}{Dx}\right]^{-1} &= \frac{(1+r^2)^3}{4(1-r^2)} \frac{2}{(1+r^2)^2} \begin{bmatrix} 1 + x^2 - y^2 & 2xy \\ 2xy & 1 - x^2 + y^2 \end{bmatrix} \\ &= \frac{1+r^2}{2(1-r^2)} \begin{bmatrix} 1 + x^2 - y^2 & 2xy \\ 2xy & 1 - x^2 + y^2 \end{bmatrix} \end{aligned}$$

With the fundamental form from formula (9.17) and the inverse Jacobi matrix just obtained plugged into equation (**??**), we calculate

$$\begin{aligned}
\begin{bmatrix} \overline{E} & \overline{F} \\ \overline{F} & \overline{G} \end{bmatrix} &= \frac{(1+r^2)^2}{4(1-r^2)^2} \frac{4}{(1-r^2)^2} \begin{bmatrix} 1+x^2-y^2 & 2xy \\ 2xy & 1-x^2+y^2 \end{bmatrix}^2 \\
&= \frac{(1+r^2)^2}{(1-r^2)^4} \begin{bmatrix} 1+2x^2-2y^2+(x^2-y^2)^2+4x^2y^2 & 4xy \\ 4xy & 1-2x^2+2y^2+r^4 \end{bmatrix} \\
&= \frac{(1+r^2)^2}{(1-r^2)^4} \begin{bmatrix} (1+r^2)^2-4y^2 & 4xy \\ 4xy & (1+r^2)^2-4x^2 \end{bmatrix}
\end{aligned}$$

This is the new fundamental form. We need still to introduce the new coordinates (X,Y). We use the short hands $r^2 = x^2+y^2$ and $R^2 = X^2+Y^2$. By means of equation (9.16) we get

$$1-X^2 = \frac{(1+r^2)^2-4x^2}{(1+r^2)^2} \quad , \quad 1-Y^2 = \frac{(1+r^2)^2-4y^2}{(1+r^2)^2} \quad ,$$
$$XY = \frac{4xy}{(1+r^2)^2} \quad , \quad 1-R^2 = \frac{(1-r^2)^2}{(1+r^2)^2}$$

Thus the new fundamental form miracously simplifies to be

$$\begin{bmatrix} \overline{E} & \overline{F} \\ \overline{F} & \overline{G} \end{bmatrix} = \frac{1}{(1-R^2)^2} \begin{bmatrix} 1-Y^2 & XY \\ XY & 1-X^2 \end{bmatrix} \tag{9.19}$$

For the line element we get from this fundamental form

$$\begin{aligned}
ds^2 &= \overline{E}\, dX^2 + 2\overline{F}\, dX dY + \overline{G}\, dY^2 \\
&= \frac{(1-Y^2)dX^2 + 2XY\, dX dY + (1-X^2)dY^2}{(1-R^2)^2} \\
&= \frac{dX^2+dY^2-(X dY - Y dX)^2}{(1-X^2-Y^2)^2}
\end{aligned}$$

□

Problem 9.5 (Gaussian curvature of the Hilbert-Klein metric). *Use Gauss' characteristic equation* (9.5) *and check directly that the Hilbert-Klein metric* (9.15) *from proposition 9.3 has constant Gaussian curvature* $K=-1$. *We use polar coordinates* $X = r\cos\theta, Y = r\sin\theta$ *and convert formula to*

$$ds^2 = \frac{dr^2 + r^2(1-r^2)d\theta^2}{(1-r^2)^2} \tag{9.20}$$

since this simplifies the calculation considerably.

Answer. We have to put $u = r$ and $v = \theta$. The first fundamental form and its coefficients become

$$ds^2 = Edr^2 + 2Fdrd\theta + Gd\theta^2$$
$$E = (1-r^2)^{-2}\,, \quad F = 0\,, \quad G = r^2(1-r^2)^{-1}$$
$$H = \sqrt{EG - F^2} = r(1-r^2)^{-3/2}$$

Hence we get for the Gaussian curvature K from the characteristic equation

$$\begin{aligned} 2HK &= -\frac{\partial}{\partial r}\left[H^{-1}\frac{\partial G}{\partial u}\right] = -\frac{\partial}{\partial r}\left[r^{-1}(1-r^2)^{3/2}\frac{\partial}{\partial r}\frac{r^2}{1-r^2}\right] \\ &= -\frac{\partial}{\partial r}\left[r^{-1}(1-r^2)^{3/2}\,\frac{2r(1-r^2)-r^2(-2r)}{(1-r^2)^2}\right] = -\frac{\partial}{\partial r}\left[2(1-r^2)^{-1/2}\right] \\ &= (-2)(-1/2)(1-r^2)^{-3/2}(-2r) = -2r(1-r^2)^{-3/2} = -2H \end{aligned}$$

We get the constant Gaussian curvature $K = -1$, as expected.

Problem 9.6 (Distortion of angles by the Hilbert-Klein metric). *We use polar coordinates to simplify the calculation, and the corresponding contravariant components for the tangent vectors. At a point K with polar coordinates $X = r\cos\theta, Y = r\sin\theta$ are attached the radial tangent vector $(a_r, a_\theta) = (1,0)$ and any other tangent vector (b_r, b_θ). Hence the apparent angle α satisfies*

$$\cos\alpha = \frac{b_r}{\sqrt{b_r^2 + r^2 b_\theta^2}} \quad \textit{and} \quad \tan\alpha = \frac{r b_\theta}{b_r}$$

Check with the Hilbert-Klein metric (9.20) *that the angle ω between the two vectors in Klein's model satisfies*

$$\tan\omega = \tan\alpha\sqrt{1-r^2} \tag{9.21}$$

Answer. The apparent angle α and the hyperbolic angle ω between the two tangent vectors (a_r, a_θ) and (b_r, b_θ) satisfy

$$\cos\alpha = \frac{a_r b_r + r^2 a_\theta b_\theta}{\sqrt{(a_r^2 + r^2 a_\theta^2)}\cdot\sqrt{(b_r^2 + r^2 b_\theta^2)}}$$
$$\cos\omega = \frac{a_r b_r + r^2(1-r^2)a_\theta b_\theta}{\sqrt{a_r^2 + r^2(1-r^2)a_\theta^2}\cdot\sqrt{b_r^2 + r^2(1-r^2)b_\theta^2}}$$

In the given example with $a_r = 1, a_\theta = 0$ the expressions simplify

$$\cos\alpha = \frac{b_r}{\sqrt{b_r^2 + r^2 b_\theta^2}} \quad \text{and} \quad \tan^2\alpha = \frac{r^2 b_\theta^2}{b_r^2}$$
$$\cos\omega = \frac{b_r}{\sqrt{b_r^2 + r^2(1-r^2)b_\theta^2}} \quad \text{and} \quad \tan^2\omega = \frac{r^2(1-r^2)b_\theta^2}{b_r^2}$$

Hence

$$\tan\omega = \tan\alpha\sqrt{1-r^2}$$

9.5 A second proof of Gauss' remarkable theorem

The most enlightened proof that the Gaussian curvature is an intrinsic property of the surface uses Gauss' notion of integral curvature. For any domain G on a given curved surface, the integral curvature is defined as the integral $\int\int_G K dA$, where dA denotes the area element of the surface.

Take a geodesic triangle $\triangle ABC$. Let T denote the region bounded by the geodesics between any three given points A, B, C on the surface. Let α, β, γ be the angles (between tangents) to the geodesics at the three vertices. Gauss proves

$$\int\int_T K dA = \alpha + \beta + \gamma - \pi \tag{9.22}$$

where angles are to be measured in radians. The quantity on the right hand side is the deviation of the angle sum $\alpha + \beta + \gamma$ from the Euclidean value $180°$, respectively π. [71] The quantity $\alpha + \beta + \gamma - \pi$ is called the excess of the triangle $\triangle ABC$. For the hyperbolic case, the excess is negative. In that case, one calculates using the excess times -1, which is called defect. In words, Gauss' theorem tells the following:

For a geodesic triangle, the integral curvature equals the excess of its angle sum.

This theorem, Gauss says, ought to be counted as a most elegant theorem.

I discuss a few immediate, but important consequences of (9.22). First of all, instead of the complicated characteristic equation (9.5), one has a simple property of a geodesic triangle from which to derive the Gaussian curvature in a limiting process. Secondly, as an immediate implications of (9.22), the Gaussian curvature is an intrinsic property of a curved surface. Recall that both geodesics, as well as measurement of area depend only on the first fundamental form. Hence, because of (9.22), the same is true for the Gaussian curvature.

Another easy consequence of (9.22) is obtained from the special case of a sphere. For this surface, the Gaussian curvature is constant, and equal to $K = R^{-2}$ where R is the radius of the sphere. Hence one obtains for the area of a spherical triangle $A = (\alpha + \beta + \gamma - \pi)R^2$. as was already known before Gauss, e.g. to Lambert.

Problem 9.7. *Tile a sphere by equilateral triangles. It can be done in three ways:*

(i) Four triangles with $\alpha = \beta = \gamma = 120°$.

(ii) Eight triangles with $\alpha = \beta = \gamma = 90°$.

(iii) N *triangles with* $\alpha = \beta = \gamma = 72°$.

Explain and draw these tilings. To which Platonic bodies do the vertices correspond? Determine the surface area of the sphere from (i) and (ii), then get the number N *in (iii).*

[71] In radian measures, the Euclidean angle sum is π.

Answer. From item (i), the area of the sphere is

$$A = 4 \cdot \left(3\frac{2\pi}{3} - \pi \right) R^2 = 4\pi R^2$$

The vertices of the four triangles form a tetrahedron. Similarly, item (ii) yields

$$A = 8 \cdot \left(3\frac{\pi}{2} - \pi \right) R^2 = 4\pi R^2$$

The vertices of the eight triangles form a octahedron.
We can now calculate the number N of triangles in the tiling (iii). Because of

$$A = N \cdot \left(3\frac{2\pi}{5} - \pi \right) = 4\pi R^2$$

one gets $N = 20$. The vertices of the twenty triangles form an icosahedron.

Here is a further important consequence of equation (9.22).

Corollary 66 (A common bound for the area of all triangles). *On a surface with negative constant Gaussian curvature $K < 0$, the area of any triangle is less than $\frac{\pi}{-K}$.*

On December 17, 1799, Gauss wrote to his friend, the Hungarian mathematician Wolfgang Farkas Bolyai (1775-1856):

> As for me, I have already made some progress in my work. However, the path I have chosen does not lead at all to the goal which we seek [deduction of the parallel axiom], and which you assure me you have reached. It seems rather to compel me to doubt the truth of geometry itself. It is true that I have come upon much which by most people would be held to constitute a proof; but in my eyes it proves as good as nothing. For example, if we could show that a rectilinear triangle whose area would be greater than any given area is possible, then I would be ready to prove the whole of Euclidean geometry absolutely rigorously.
>
> Most people would certainly let this stand as an axiom; but I, no! It would indeed be possible that the area might always remain below a certain limit, however far apart the three angular points of the triangle were taken.

From about 1813 on Gauss developed his new geometry. He became convinced that it was logically consistent and rather sure that it might be applicable. His letter written in 1817 to Olbers says:

I am becoming more and more convinced that the physical necessity of our Euclidean geometry cannot be proved, at least not by human reason nor for human reason. Perhaps in another life we will be able to obtain insight into the nature of space, which is now unattainable. Until then we must place geometry not in the same class with arithmetic, which is purely a priori, but with mechanics.

Problem 9.8. **(a)** *Find two further enlightening statements of Gauss, and comments on all four statements.*

(i) Are they courageous?

(ii) Are they to the benefit of the scientific community?

(iii) Are they helpful for the person he addresses?

(iv) Are they just against other people?

(v) What would you have done in Gauss' place?

(b) *Choose two of Gauss' comments. Write a letter as you imagine you would have written in place of Gauss.*

Problem 9.9. *To test the applicability of Euclidean geometry and his non-Euclidean geometry, Gauss actually measured the sum of the angles of the triangle formed by three mountain peaks in middle Germany: Broken, Hohenhagen, and Inselberg. the sides of the triangle were* $69, 85, 197$ km*. His measurement yielded that the angle sum exceeded* 180° *by* $14".85$*.*

(a) *Use Herons formula* $A = \sqrt{s(s-a)(s-b)(s-c)}$ *to calculate the area of the triangle, in a very good approximation.*

(b) *Take* $R = 6378$ km *as radius of the earth. Calculate the angle excess for a* <u>*spherical*</u> *triangle between the three mountain peaks. You need to convert angular measurements!* 1 *radian equals* $\dfrac{180^\circ}{\pi} = \dfrac{180 \cdot 3600"}{\pi}$.

(c) *Is the triangle that Gauss measured actually a spherical triangle. Why or why not?*

(d) *Reflect on the motives why Gauss did his measurement. Find and read some further sources. Think of the following and further motives and possibilities. Did Gauss really just want to*

(i) check accuracy?

(ii) check geometry?

(iii) It was just a theoretical thought experiment, not really performed.

Answer. **(a)** Herons formula give the area $A = \sqrt{s(s-a)(s-b)(s-c)} = 2929.42\ \text{km}^2$.

(b) Take $R = 6378$ km as radius of the earth. The angle excess for a <u>spherical</u> triangle between the three mountain peaks is

$$\alpha + \beta + \gamma - \pi = \frac{A}{R^2} = 7.201\, 10^{-5} \tag{9.23}$$

This is the value in radian measure. Converted to degrees, we get .00413° which is 14.9".

(c) Of course the triangle one measures is not a spherical triangle, since light rays do not follow the curvature of the earth.

9.6 Principal and Gaussian curvature of rotation surfaces

Before introducing the pseudo-sphere, we need some facts about the curvature of general rotation surfaces. We take the graph of an arbitrary function $y = f(x)$, and rotate it about the y-axis to produce a rotation surface in three dimensional space. The first principle curvature of a rotation surface in the xy plane is

$$\kappa_1 = \frac{y''}{(1 + y'^2)^{\frac{3}{2}}} \tag{9.24}$$

This is just the curvature of the graph $y = f(x)$.

Recall that the perpendicular to the tangent of a curve is called the normal of the curve. The second principal curvature occurs for a section of the surface by a plane $\mathcal{P}_2$, which intersects the xy plane along the normal of the curve $y = f(x)$, and is perpendicular to the xy plane. The second principal curvature is

$$\kappa_2 = \frac{y'}{x(1 + y'^2)^{\frac{1}{2}}} \tag{9.25}$$

Proposition 9.4. *The Gaussian curvature of the rotation surface produced by rotating the graph of $y = f(x)$ around the y-axis, is the product*

$$K = \frac{y'\, y''}{x(1 + y'^2)^2} \tag{9.26}$$

The formula (9.24) for a curvature of a plane curve is standard. Finally, since $K = \kappa_1 \kappa_2$, formulas (9.24) and (9.25) imply the claim (9.26).

Here is an argument to justify (9.25): Let $\tan\beta = y'$ be tangent of the slope angle for $y = f(x)$, as usual. Calculate $\sin\beta$. Calculate the hypothenuse AB of the right $\triangle ABC$, with vertex $A = (x, y)$ on the curve, leg AC parallel to the x-axis, leg BC on the axis of rotation, and hypothenuse AB perpendicular to the curve. One can show that point B is the center of the best approximating circle in the plane $\mathcal{P}_2$. Hence $\kappa_2 = \frac{1}{AB}$. Use this idea to get the main curvature κ_2.

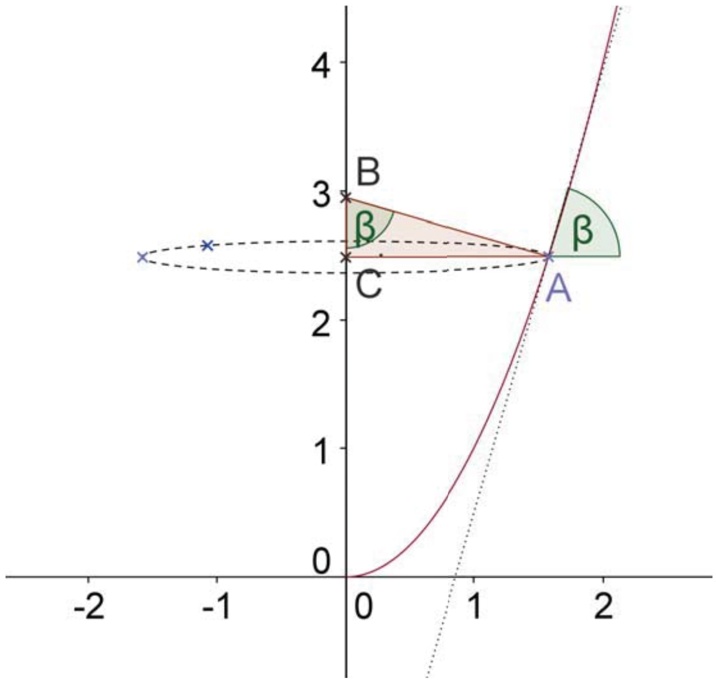

Figure 9.1: Curvature of a rotation surface.

Answer. $\tan\beta = \frac{\overline{AC}}{\overline{BC}} = y'$ Hence

$$\sin\beta = \frac{\overline{AC}}{\overline{AB}} = \frac{\overline{AC}}{\sqrt{\overline{BC}^2 + \overline{AC}^2}} = \frac{y'}{\sqrt{1+y'^2}}$$

$$\kappa_2 = \frac{1}{\overline{AB}} = \frac{\sin\beta}{x} = \frac{y'}{x(1+y'^2)^{\frac{1}{2}}}$$

A second proof of Proposition 9.4 . On the surface of rotation, we choose as parameters $u = \phi$ the rotation angle, and $v = r$ the distance from the rotation axis. Since the y-axis is the axis of rotation, the surface of rotation gets the parametric representation

$$x = v\cos u$$
$$y = f(v)$$
$$z = v\sin u$$

The derivatives by the parameters are

$$\begin{aligned} x_u &= -v\sin u & x_v &= \cos u \\ y_u &= 0 & y_v &= f'(v) \\ z_u &= v\cos u & z_v &= \sin u \end{aligned} \tag{9.27}$$

From these derivatives, one gets the first fundamental form. We use the general formulas

$$E = x_u^2 + y_u^2 + z_u^2$$
$$F = x_u x_v + y_u y_v + z_u z_v \tag{9.3}$$
$$G = x_v^2 + y_v^2 + z_v^2$$

valid for any surface, and specialize to the surface of rotation given above. Now calculate E, F, G from (9.3) and (9.27), to get the first fundamental form (9.2). Next get the root of the determinant $H = \sqrt{EG - F^2}$. Finally calculate the Gaussian curvature from the characteristic equation (9.5). □

Problem 9.10. *Use the approach as indicated to confirm formula* (9.26).

Answer. One gets $E = v^2$, $F = 0$ and $G = 1 + f'^2$ and

$$ds^2 = v^2 du^2 + (1 + f'^2) dv^2 = r^2 d\phi^2 + (1 + f'(r)^2) dr^2$$

The root of the determinant is $H = v\sqrt{1 + f'^2}$. Because all four quantities E, F, G, H depend only on v, the partial derivatives by u all vanish. Thus Gauss' characteristic equation (9.5) can be simplified to yield

$$\begin{aligned} K &= \frac{1}{2H}\frac{\partial}{\partial u}\left[\frac{F}{EH}\frac{\partial E}{\partial v} - \frac{1}{H}\frac{\partial G}{\partial u}\right] + \frac{1}{2H}\frac{\partial}{\partial v}\left[\frac{2}{H}\frac{\partial F}{\partial u} - \frac{1}{H}\frac{\partial E}{\partial v} - \frac{F}{EH}\frac{\partial E}{\partial u}\right] \\ &= \frac{1}{2H}\frac{\partial}{\partial v}\left[-\frac{1}{H}\frac{\partial E}{\partial v}\right] = -\frac{1}{2H}\frac{d}{dv}\left[\frac{2v}{H}\right] \end{aligned}$$

Now use $H = v(1 + f'^2)^{\frac{1}{2}}$ and go on. We arrive at

$$\begin{aligned} K &= -\frac{1}{2H}\frac{d}{dv}\left[\frac{2v}{H}\right] = -\frac{1}{v\sqrt{1 + f'^2}}\frac{d(1 + f'^2)^{-\frac{1}{2}}}{dv} \\ &= \frac{1}{2v\sqrt{1 + f'^2}}\left[(1 + f'^2)^{-\frac{3}{2}}\right] 2f'f'' = \frac{f'f''}{v(1 + f'^2)^2} \end{aligned}$$

This result is equivalent to formula (9.26), since $x = v = r$ is the distance from the axis of rotation and $y = f(x)$.

Problem 9.11. *Calculate the Gaussian curvature of a three dimensional sphere of radius a.*

Answer. The sphere is provided by rotating the graph of $x^2 + y^2 = a^2$ about the y-axis. Implicit differentiation yields $2x + 2yy' = 0$ and hence

$$\begin{aligned} y' &= -\frac{x}{y} \\ y'' &= -\frac{1 \cdot y - xy'}{y^2} = \frac{xy' - y}{y^2} = \frac{-x^2/y - y}{y^2} = -\frac{a^2}{y^3} \\ K &= \frac{y'y''}{x(1 + y'^2)^2} = \frac{a^2}{y^4\,(1 + x^2/y^2)^2} = \frac{a^2}{(x^2 + y^2)^2} = \frac{1}{a^2} \end{aligned}$$

9.7 The pseudo-sphere

The issue is now to find a rotation surface of constant negative Gaussian curvature $K = -a^{-2}$. Such a surface is called pseudo-sphere.

Problem 9.12. *Use the formula* (9.26) *for the Gaussian curvature, and get a differential equation of first order for the function* $u := y'^2$. *You may begin by getting the derivative* $\frac{du}{dx}$.

Answer. The derivative of the function $u := y'^2$ is $\frac{du}{dx} = 2y'y''$. Next, I put the requirement $K = -a^{-2}$ into the formula (9.26). One gets

$$\frac{y'\, y''}{x(1+y'^2)^2} = -\frac{1}{a^2} \tag{9.28}$$

$$2y'\, y'' = -\frac{2x(1+y'^2)^2}{a^2} \tag{9.29}$$

$$\frac{du}{dx} = -\frac{2x(1+u)^2}{a^2} \tag{9.30}$$

Problem 9.13. *Solve the differential equation*

$$\frac{du}{dx} = -\frac{2x(1+u)^2}{a^2} \tag{9.30}$$

by separation of variable. For simplicity, we use the initial data $u(a) = 0$*, and get a curve through the point* $x = a, u = 0$.

Answer.

$$\begin{aligned}
\frac{du}{dx} &= -\frac{2x(1+u)^2}{a^2} \\
\frac{du}{(1+u)^2} &= -\frac{2xdx}{a^2} \\
\int_a^u \frac{du}{(1+u)^2}\, du &= -\int_0^x \frac{2xdx}{a^2}\, dx \\
\left[-\frac{1}{1+u}\right]_a^u &= \left[-\frac{x^2}{a^2}\right]_0^x \\
-\frac{1}{1+u} + 1 &= -\frac{x^2}{a^2} + 1 \\
1 + u &= \frac{a^2}{x^2} \\
u &= \frac{a^2 - x^2}{x^2}
\end{aligned}$$

Problem 9.14. *Check that the differential equation*

$$y' = -\frac{\sqrt{a^2 - x^2}}{x}$$

with $|x| \le a$*, has the solution*

$$y = a \ln \left[\frac{a + \sqrt{a^2 - x^2}}{x} \right] - \sqrt{a^2 - x^2} + C \tag{9.31}$$

Too, find the general solution of the equation

$$y' = +\frac{\sqrt{a^2 - x^2}}{x}$$

Answer.

$$y = a \ln \left[\frac{a - \sqrt{a^2 - x^2}}{x} \right] + \sqrt{a^2 - x^2} + C$$

Definition 9.1 (Pseudo-sphere). The rotation surface with constant negative Gaussian curvature is called a *pseudo-sphere*. With curvature $K = -a^{-2}$ and the y-axis as axis of rotation, its equation is

$$y = a \ln \left[\frac{a + \sqrt{a^2 - x^2 - z^2}}{\sqrt{x^2 + z^2}} \right] - \sqrt{a^2 - x^2 - z^2}$$

Problem 9.15. *Check, once more, that the Gaussian curvature of the specified surface is* $K = -a^{-2}$.

Answer.

Problem 9.16. *Check the following fact: The segment on the tangent to the curve* (9.31)*, between the touching point* T*, and the intersection* S *of the tangent with the* y*-axis has always the same length* a*. For that reason, the curve* (9.31) *is called* <u>*tractrix*</u>.

Answer. Take the right $\triangle TSC$, formed by the segment TS on the tangent at point T, and the perpendicular from T onto the y-axis. [72] We know that

$$y' = \tan \alpha = \frac{\overline{SC}}{\overline{TC}}$$
$$\overline{TS}^2 = \overline{TC}^2 + \overline{SC}^2 = x^2(1 + y'^2) = a^2$$

[72]This triangle is different from triangle $\triangle ABC$ in the figure on page 915.

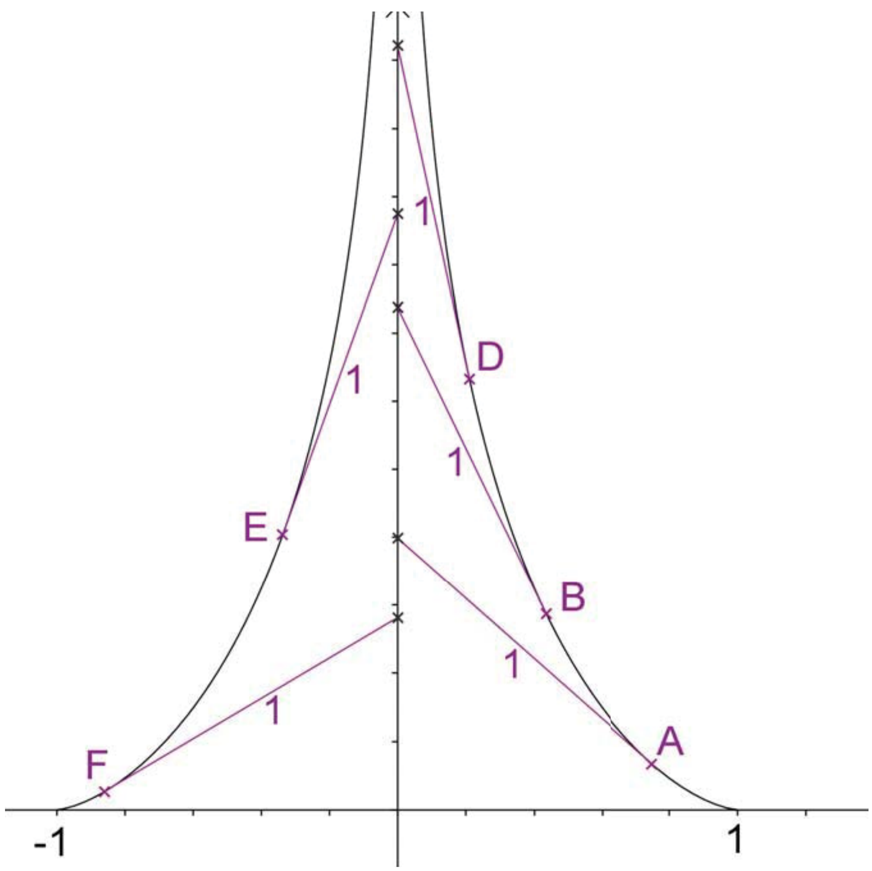

Figure 9.2: The tractrix has a segment on its tangent of constant length.

Problem 9.17. *The surface area of of rotation surface, made by rotating $y = f(x)$ about the y-axis, for $x_1 \le x \le x_2$ is*

$$S = \int_{x_1}^{x_2} 2\pi x \sqrt{1 + y'^2}\, dx$$

Calculate the surface of the pseudo-sphere for bounds $0 < x \le a$.

Answer. Because of $1 + y'^2 = 1 + u = \frac{a^2}{x^2}$, we get

$$S = \int_0^a 2\pi x \, \frac{a}{x}\, dx = 2\pi a^2$$

We introduce now (ϕ, r) as two convenient coordinates on the pseudo-sphere. As first coordinate, we choose the angle of rotation ϕ about the y-axis. The second coordinate is the radius $r = \sqrt{x^2 + z^2}$ measured from the axis of rotation, Up to now it was called x, but now I choose to name it r. The three parameters r, ϕ, y are cylindrical coordinates of three dimensional space. The first two of them are convenient parameters on the pseudo-sphere.

Proposition 9.5 (Riemann Metric for the Pseudo Sphere). *The infinitesimal distance ds of points with coordinates (ϕ, r) and $(\phi + d\phi, r + dr)$ is*

$$ds^2 = r^2 d\phi^2 + \frac{a^2}{r^2} dr^2 \tag{9.32}$$

Proof. The distance on the pseudo-sphere is calculated from the usual Euclidean distance for points of the three dimensional space into which the surface is embedded. At first, I convert the distance from Cartesian to cylindrical coordinates. Because the y-axis is the rotation axis, its coordinate stays, but the pair (x, z) is converted to polar coordinates. Hence one gets

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\phi^2 + dy^2 \tag{9.33}$$

We restrict to points on the pseudo-sphere. Hence the coordinates r and y are related in the same way as x and y before. Thus $y' = -\frac{\sqrt{a^2 - x^2}}{x}$ gets

$$\frac{dy}{dr} = -\frac{\sqrt{a^2 - r^2}}{r} \tag{9.34}$$

Now we use (9.34) to eliminate y from (9.33) and get

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\phi^2 + dy^2 = dr^2 + r^2 d\phi^2 + \left(\frac{dy}{dr}\right)^2 dr^2 \\ &= dr^2 + r^2 d\phi^2 + \frac{a^2 - r^2}{r^2} dr^2 = r^2 d\phi^2 + \frac{a^2}{r^2} dr^2 \end{aligned}$$

as to be shown. As an alternative, we can use the first fundamental form calculated above. Since $1+f'(r)^2 = 1+\frac{a^2-r^2}{r^2} = \frac{a^2}{r^2}$, one gets again

$$ds^2 = r^2 d\phi^2 + (1+f'(r)^2)dr^2 = r^2 d\phi^2 + \frac{a^2}{r^2}dr^2$$

□

9.8 Poincaré half-plane and Poincaré disk

Throughout, we denote the upper open half-plane by $H = \{(u,v) : v > 0\}$. Its boundary is just the real axis $\partial H = \{(u,v) : v = 0\}$. The open unit disk is denoted by $D = \{z = x+iy : x^2+y^2 < 1\}$, and its boundary is $\partial D = \{z = x+iy : x^2+y^2 = 1\}$. The following isometric mapping of the half-plane to the disk is used in this section. It differs from the one used in the previous section by a rotation of the disk by a right angle. We repeat for convenience.

Proposition 9.6 (Isometric Mapping of the Half-plane to the Disk). *The linear fractional function*

$$z = \frac{iw+1}{w+i} \tag{9.35}$$

is a conformal mapping and a bijection from $\mathbf{C} \cup \{\infty\}$ *to* $\mathbf{C} \cup \{\infty\}$*. The inverse mapping is*

$$w = \frac{1-iz}{z-i} \tag{9.36}$$

These mappings preserves angles, the cross ratio, the orientation, and map generalized circles to generalized circles.

The upper half-plane $H = \{w = u+iv : v > 0\}$ *is mapped onto the unit disk* $D = \{z = x+iy : x^2+y^2 < 1\}$*. Especially*

$$w = 0 \mapsto z = -i\,, \quad w = 1 \mapsto z = 1\,, \quad w = \infty \mapsto z = i\,, \quad w = i \mapsto z = 0$$

Proposition 9.7 (Riemann Metric for Poincaré's half-plane). *In the Poincaré half-plane, the infinitesimal hyperbolic distance* ds *of points with coordinates* (u,v) *and* $(u+du, v+dv)$ *is*

$$(ds_H)^2 = \frac{du^2+dv^2}{v^2} \tag{9.37}$$

The mapping (9.35) *provides an isometry between the half-plane and the disk:*

$$ds_D = ds_H \tag{7.5}$$

Proof. The metric of the half plane is calculated from the known metric

$$(ds_D)^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} \tag{9.14}$$

of the Poinaré disk model. The mapping

$$z = \frac{iw + 1}{w + i} \tag{9.35}$$

provides an isometry from the half-plane to the disk. The denominator is

$$1 - |\, z|^2 = \frac{(w+i)(\overline{w} - i) - (iw+1)(-i\overline{w} + 1)}{|w+i|^2} = \frac{2i\overline{w} - 2iw}{|w+i|^2} = \frac{4v}{|w+i|^2}$$

The derivative of the mapping (9.35) is

$$\frac{dz}{dw} = -\frac{2}{(w+i)^2}$$

Putting the last two formulas into (9.14) yields

$$\begin{aligned} ds^2 &= \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} = \frac{4|\, dz|^2}{(1 - |z|^2)^2} \\ &= 4 \left| \frac{dz}{dw} \right|^2 |\, dw|^2 \left(\frac{|w+i|^2}{4v} \right)^2 = 4 \left| \frac{2}{(w+i)^2} \right|^2 |\, dw|^2 \left(\frac{|w+i|^2}{4v} \right)^2 \\ &= \frac{|\, dw|^2}{v^2} = \frac{du^2 + dv^2}{v^2} \end{aligned}$$

Hence formula (9.37) arises from the isometry (9.35) between half-plane and the disk. □

9.9 Embedding the pseudo-sphere into Poincaré's half-plane

Proposition 9.8. *The mapping*

$$w = \phi + i\, \frac{a}{r} \tag{9.38}$$

transforms the line element $ds_{\mathcal{PS}}$ of the pseudo-sphere to the line element ds_H of the half-plane such that

$$ds_{\mathcal{PS}} = a\,(ds_H) \tag{9.39}$$

For $a = 1$, we get an isometry. This is just the case with Gaussian curvature $K = -a^{-2} = -1$. Because an isometry conserves the Gaussian curvature, this shows that the Poincaré half-plane has Gaussian curvature -1.

Proof. We separate equation (9.38) into its real- and imaginary part to get

$$u = \phi\,, \quad v = \frac{a}{r} \tag{9.40}$$

Using its derivatives, we plug into

$$ds_H{}^2 = \frac{du^2 + dv^2}{v^2} \tag{9.37}$$

One gets

$$ds_H{}^2 = \frac{d\phi^2 + \left(-ar^{-2}\right)^2 dr^2}{a^2 r^{-2}} = a^{-2}\left(r^2 d\phi^2 + \frac{a^2}{r^2} dr^2\right)$$

Now comparing with

$$ds_{\mathcal{P}S}^2 = r^2 d\phi^2 + \frac{a^2}{r^2} dr^2 \tag{9.32}$$

one concludes

$$ds_H{}^2 = a^{-2}(ds_{\mathcal{P}S})^2$$

and hence equation (9.39) holds. □

9.10 Embedding the pseudo-sphere into Poincaré's disk

The next goal is to construct an isometric mapping from the pseudo-sphere to the Poincaré disk. It is convenient to get this mapping as composition of a mapping from the pseudo-sphere to the half-plane, and the conformal mapping from the half-plane to the disk.

Proposition 9.9. *We take a pseudo-sphere with $a = 1$. This normalizes the Gaussian curvature to be $K = -1$. The mapping*

$$z = \frac{r - 1 + ir\phi}{r\phi + i(r+1)} \tag{9.41}$$

maps the pseudo-sphere isometrically <u>into</u> the Poincaré disk.

Proof. The mapping (9.41) is constructed as a composition of two mappings $\mathcal{P}S \mapsto H \mapsto D$. Take the mapping $\mathcal{P}S \mapsto H$ given by equation (9.42), and the mapping $H \mapsto D$ given by equation (9.35). The composition of the mapping

$$w = \phi + i\frac{1}{r} \tag{9.42}$$

from the pseudo-sphere to the Poincaré half-plane, with the mapping

$$z = \frac{iw + 1}{w + i} \tag{9.35}$$

from the Poincaré half-plane to the Poincaré disk is the required mapping. following mapping

$$z = \frac{i\left(\phi + \frac{i}{r}\right) + 1}{\phi + \frac{i}{r} + i} = \frac{r - 1 + ir\phi}{r\phi + i(r+1)}$$

from the pseudo-sphere to the Poincaré disk. Both mappings (9.42) and (9.35) are isometries, as stated by formulas (9.39) with $a = 1$ and formula (**??**). Hence their composition (9.41) conserves the line element: $ds_{\mathcal{PS}} = ds_H = ds_D$. □

9.11 About circle-like curves

We now go back to the Poincaré disk model. At first, here are a few remarks about circle-like curves. In hyperbolic geometry, there exist three different types of circle-like curves. I define as a circle-like curve a curve which appears to be a circle in the Poincaré model.

Recall that ∂D is the boundary circle of the Poincaré disk. Take any second circle $\mathcal{C}$. I call its Euclidean center M the quasi-center. The meaning of $\mathcal{C}$ for the hyperbolic geometry of the Poincaré disk depends on the nature of the intersection of the two circles $\mathcal{C}$ and ∂D. There are three important cases:

(i) The circle $\mathcal{C}$ lies totally in the interior D. In that case, it is a circle for hyperbolic geometry. This circle has a center A in hyperbolic geometry. Note that the quasi-center M is different from the center A of $\mathcal{C}$ as an object of hyperbolic geometry.

(ii) The circle $\mathcal{C}$ touches the boundary ∂D from inside, say at endpoint E. In that case, it is a horocycle for hyperbolic geometry. A horocycle has no hyperbolic center, instead it contains an ideal point E. Hence it is unbounded. The hyperbolic circumference of a horocycle is infinite, as follows from part (c) below.

(iii) The circle $\mathcal{C}$ intersects the boundary ∂D at two endpoints E and F. In that case, the circular arc inside the disk D is either an equidistance line or a geodesic for hyperbolic geometry. A geodesic intersects ∂D perpendicularly. In the case of non perpendicular intersection of ∂D and $\mathcal{C}$, one gets an equidistance line. Actually all points of that equidistance line have the same distance from the hyperbolic straight line with ends E and F.

Problem 9.18. *Take points $Y_+ = (1, 0)$ and $O = (0, 0)$. Find the analytic equation for a horocycle $\mathcal{H}$ with apparent diameter OY_+.*

Answer. In complex notation, point Y_+ is i. The quasi-center is $M = \frac{i}{2}$, and the apparent radius is $\frac{1}{2}$. Hence one gets the equation

$$\left|z - \frac{i}{2}\right|^2 = \frac{1}{4}$$

$$x^2 + \left(y - \frac{1}{2}\right)^2 - \frac{1}{4} = 0$$

$$x^2 + y(y-1) = 0$$

We need another more convenient parametric equation for the horocycle $\mathcal{H}$. Let $Z = (x, y)$ be any point on $\mathcal{H}$ and define the circumference angle $\rho \cong \angle OY_+Z$. Calculate $\tan\rho$ in terms of (x, y). Then express x and y in terms of the central angle $2\rho \cong \angle OMZ$. Use double angle formulas, and finally express x and y in terms of $\tan\rho$.

Answer.

$$
\begin{aligned}
\tan\rho &= \frac{x}{1-y} = \frac{y}{x} \\
x &= \frac{\sin 2\rho}{2} = \sin\rho\cos\rho = \frac{\tan\rho}{1+\tan^2\rho} \\
y &= \frac{1-\cos 2\rho}{2} = \sin^2\rho = \frac{\tan^2\rho}{1+\tan^2\rho}
\end{aligned}
\tag{9.43}
$$

Problem 9.19. *Confirm that the hyperbolic arc length of the arc OZ on the horocycle $\mathcal{H}$ is just $s = 2\tan\rho$.*

Answer. Let $t = \tan\rho$. Differentiation yields

$$
\begin{aligned}
x &= \frac{t}{1+t^2}\,, & \frac{dx}{dt} &= \frac{1-t^2}{(1+t^2)^2} \\
y &= \frac{t^2}{1+t^2}\,, & \frac{dy}{dt} &= \frac{2t}{(1+t^2)^2} \\
1-x^2-y^2 = 1-y &= \frac{1}{1+t^2} & \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= \frac{1}{(1+t^2)^2}
\end{aligned}
$$

Hence the hyperbolic metric (9.14) implies

$$
\left(\frac{ds}{dt}\right)^2 = 4(1-x^2-y^2)^{-2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] = 4
$$

Hence by elementary integration $s = 2t = 2\tan\rho$.

Problem 9.20. *Give the representation of the horocycle $\mathcal{H}$ with this arc length as parameter. Explain in a drawing, how to measure the arc length on this horocycle.*

Answer. We get the parametrization

$$
\begin{aligned}
x &= \frac{2s}{4+s^2} \\
y &= \frac{s^2}{4+s^2}
\end{aligned}
\tag{9.44}
$$

The hyperbolic arc length of OZ on the horocycle $\mathcal{H}$ is the Euclidean length $|Y_-Z'|$ since $s = 2\tan\rho = |Y_-Z'|$.

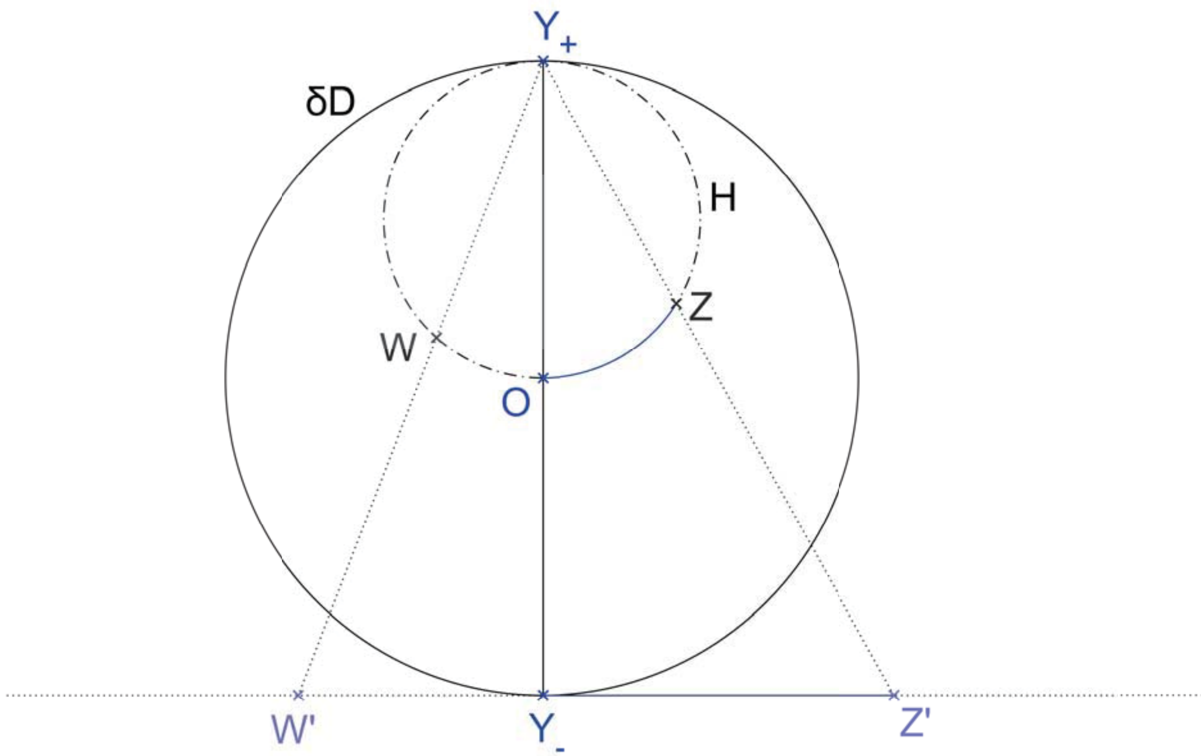

Figure 9.3: Measuring an arc of a horocycle.

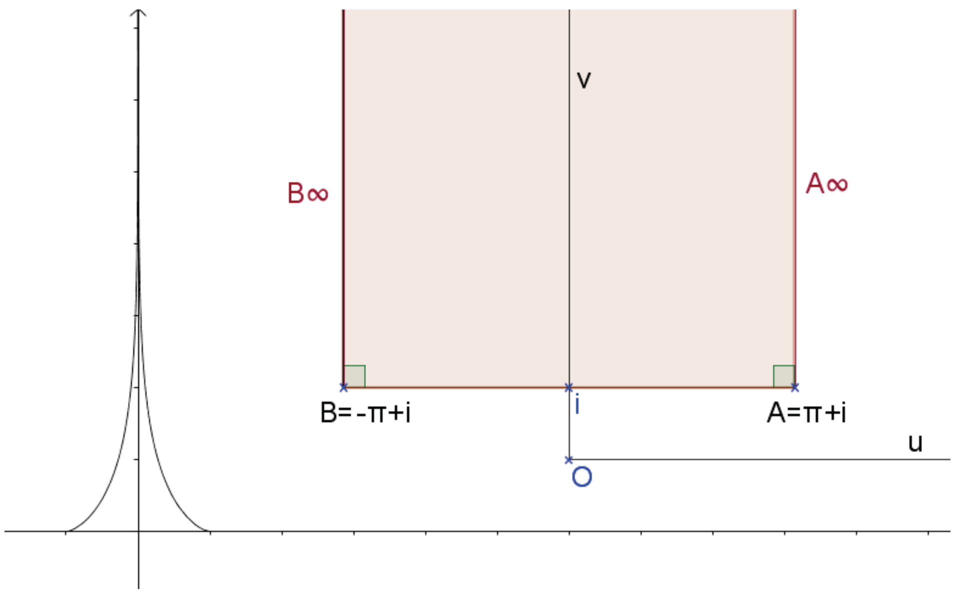

Figure 9.4: Isometry of the sliced pseudo-sphere to a half infinity strip.

9.12 Mapping the boundaries

There cannot exist an isometry of between the pseudo-sphere and the half-plane, since they have different topologies.

A corresponding problem already arises in Euclidean geometry, for the construction of for an isometry between the cylinder and the plane. At least, there exists a non-invertible homomorphism from the plane onto the cylinder. This homomorphism can be restricted to on isomorphism between a strip of the plane and the *sliced cylinder*.

We return to the hyperbolic case. By slicing the pseudo-sphere, we get an isomorphism of the sliced pseudo-sphere into a strip of the half-plane, and furthermore into part of the disk, too.

The pseudo-sphere is sliced along the geodesic in the negative (x, y)-plane, restricting the rotation angle to the half-open interval $-\pi \le \phi < \pi$. The mapping

$$w = \phi + i\,\frac{1}{r} \tag{9.42}$$

maps the sliced pseudo-sphere onto a half open rectangular domain $\mathcal{PS}_H$ in the upper half-plane. The boundary of $\mathcal{PS}_H$ consists of a segment AB with the endpoints $A = -\pi + i, B = \pi + i$, and two unbounded rays $\overrightarrow{A\infty}$ and $\overrightarrow{B\infty}$ with vertices A and B parallel to the positive v axis. Furthermore, we map the sliced pseudo-sphere to the Poincaré disk via the isometry (9.41). The image $\mathcal{PS}_D$ of the pseudo-sphere is a part of the interior of the horocycle $\mathcal{H}$ with apparent diameter 0 to i.

Problem 9.21. *On the pseudo-sphere, we use as parameters the cylindrical coordinates r and ϕ. The boundary of the sliced pseudo-sphere is given by $r = 1$, and $-\pi < \phi < \pi$. To which curve in the disk D is the boundary mapped by the isometry* (9.41)*?*

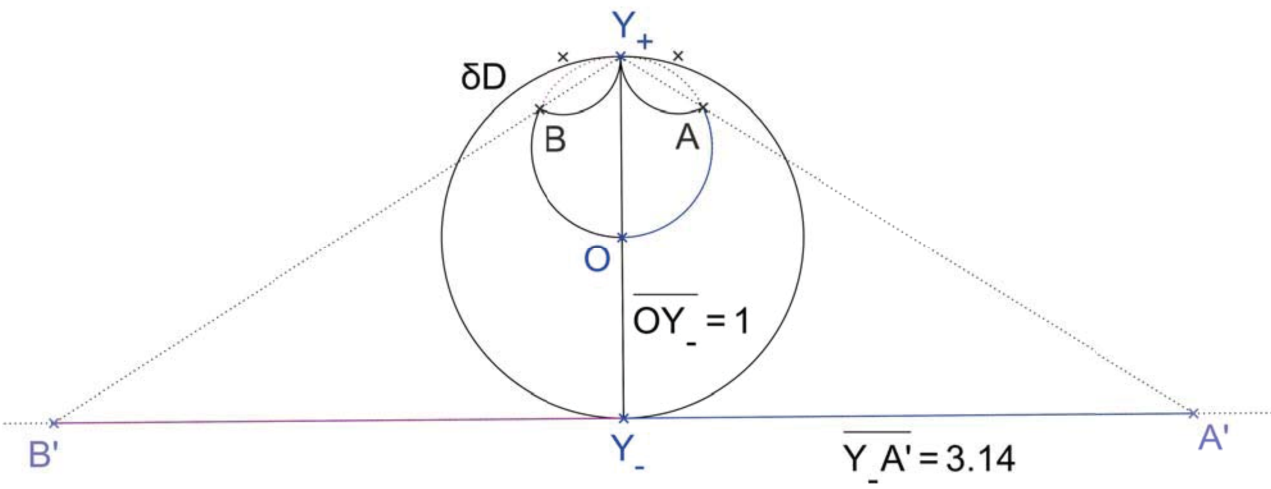

Figure 9.5: Isometric image of the sliced pseudo-sphere in the Poincaré disk.

Answer. The boundary $\partial\mathcal{P}S_D$ of image $\mathcal{P}S_D$ consists of three circular arcs. A segment AB of a horocycle $\mathcal{H}$ with endpoints

$$A = \frac{i\pi}{\pi + 2i}\,, \quad B = \frac{i\pi}{\pi - 2i}$$

as well as two geodesic rays $\overrightarrow{AY_+}$ and $\overrightarrow{BY_+}$ with vertices A and B pointing to the ideal endpoint $Y_+ = i$.

Problem 9.22. *Give a parametric equation for the boundary, with parameter ϕ, at first in complex notation for $z = x + iy$. Then separate into real and imaginary parts to get equations for x and y.*

Answer. Simply put $r = 1$ into equation (9.41). One gets

$$z = \frac{i\phi}{\phi + 2i}$$

To separate real and imaginary parts, one needs a make the denominator real:

$$z = \frac{i\phi}{\phi + 2i} = \frac{i\phi(\phi - 2i)}{(\phi + 2i)(\phi - 2i)} \tag{9.45}$$

$$x + iy = \frac{i\phi^2 + 2\phi}{\phi^2 + 4} \tag{9.46}$$

$$x = \frac{2\phi}{\phi^2 + 4} \tag{9.47}$$

$$y = \frac{\phi^2}{\phi^2 + 4} \tag{9.48}$$

Problem 9.23. *Check that your parametric equation is a circle with center $\frac{i}{2}$.*

Answer. This is a parametric equation of a circle with center $\frac{i}{2}$ because

$$z - \frac{i}{2} = \frac{i\phi}{\phi + 2i} - \frac{i}{2} = \frac{i(\phi - 2i)}{2(\phi + 2i)}$$
$$\left| z - \frac{i}{2} \right| = \frac{1}{2}$$

(d) Compare (9.46) with the result

$$x = \frac{\tan\rho}{1 + \tan^2\rho}$$
$$y = \frac{\tan^2\rho}{1 + \tan^2\rho}$$

and check that $\phi = 2\tan\rho$. Hence, because of $s = 2\tan\rho$ was shown, one concludes that $\phi = s = 2\tan\rho$. Of course $\phi = s$ follows directly because of the isometries $\mathcal{P}S \mapsto H \mapsto D$.

Problem 9.24. *Draw sketches of the pseudo-sphere as it appears in the domains $\mathcal{P}S \mapsto H \mapsto D$. Use different colors for the different vertices and edges of the boundary, but same colors for corresponding objects in all three domains $\mathcal{P}S \mapsto H \mapsto D$, as they are mapped by our isometries from above.*

Problem 9.25. *The Poincaré disk can be tiled with congruent triangles. Indeed, there exist infinitely many different types of such tilings. I choose a tiling with congruent equilateral triangles, such that at each vertex seven triangles meet. Use Gauss' remarkable theorem to calculate the hyperbolic area of one such triangle.*

Answer. I measure angles in radian measure. The angles of one triangle are all

$$\alpha = \beta = \gamma = \frac{2\pi}{7}$$

and hence the defect of the angle sum is

$$\alpha + \beta + \gamma - \pi = \frac{6\pi}{7} - \pi = -\frac{\pi}{7}$$

Since the Gaussian curvature of Poincaré's model is $K = -1$, the area is just the negative of the excess, (this is also called the defect) and is $\frac{\pi}{7}$.

Problem 9.26. *Overlay two drawings, of the tiling by equilateral triangles, and a second drawing of the image $\mathcal{P}S_D$ of the pseudo-sphere and its boundary $\partial\mathcal{P}S_D$.*

How many of those triangles of the tiling fit entirely into $\mathcal{P}S_D$?

How many triangles make up (with bids and pieces!) the total area of $\mathcal{P}S_D$?

Answer. Using item 12, one calculated that the total area of the pseudo-sphere is 2π. This equals the area of 14 triangles.