12 A Simplified Axiomatic system of Geometry—my own Suggestion

12.1 A simplified axiomatization of geometry

Introduction

The axiomatic system of Hilbert shows exceptional insight. On the other hand, it is unnatural in many respects. My simplification restricts attention to two dimensions, introduces circles in the natural classical way, and continuity in the way directly related to proofs. My goal is to give a logical analysis of the natural way to do geometry, and still keep up the level of rigor introduced by Hilbert. But I do not try to minimize the axioms as strictly as Hilbert. Especially the axioms dealing with the order of three points on a line, the SAS congruence, and the angle sum can be stated equivalently, but making less assumptions.

0. Undefined elements and relations

Elements:

- A class of undefined objects called points, denoted by $A, B, C, \ldots$
- A class of undefined objects called lines, denoted by $a, b, c, \ldots$
- A class of undefined objects called circles, denoted by $C, D, \ldots$

Relations:

- Incidence (being incident, lying on, containing)
- Order (lying between) (for points on a line)
- Congruence (for segments)
- Center (for circle)

I. Axioms of incidence

I.1 For two points $A$ and $B$ there exists a line that contains both points.

I.2 For two different points $A$ and $B$ there exists no more than one line that contains both points.

I.3a Any line contains at least two points.

I.3b There exist at least three points that do not lie on a line.

Remark. I did separate axiom I.3 into I.3a and I.3b, in order to stress there is no direct logical connection between the two sentences intended.
II. Axioms of order

II.1 If a point \(B\) lies between a point \(A\) and a point \(C\), then the points \(A, B, C\) are three distinct points of a line, and \(B\) lies between \(C\) and \(A\).

**Definition 12.1 (Segment).** Let \(A\) and \(B\) be two distinct points. The segment \(AB\) is the set consisting of the points \(A\) and \(B\) and all points lying between \(A\) and \(B\). The points \(A\) and \(B\) are called the endpoints of the segment, they are assumed to be different. The points between the endpoints are called the interior points of the segment, and the remaining points on the line \(AB\) are called the exterior points of the segment.

II.2∗ Of any three different points on a line exactly one lies between the two others.

II.3∗ For two points \(A\) and \(B\), the ray \(\overrightarrow{AB}\) contains a point \(C\) not lying inside the segment \(AB\).

**Definition 12.2 (Ray).** Given two distinct points \(A\) and \(B\), the ray \(\overrightarrow{AB}\) is the set consisting of the points \(A\) and \(B\), the points inside the segment \(AB\), and all points \(P\) on the line \(AB\) such that point \(B\) lies between vertex \(A\) and \(P\). The point \(A\) is called the vertex of the ray.

II.4 (Pasch’ Axiom) Given is a triangle \(\triangle ABC\) and a line \(a\) which does not meet any of its vertices \(A, B, C\). If the line \(a\) passes through a point of the side \(AB\), it also passes through a point of either one of the two other sides \(AC\) or \(BC\).

III. Axioms for circles

III.1 For any center \(O\) and point \(A \neq O\), there exists a unique circle with center \(O\) through the point \(A\).

III.2 A circle \(C\) and a ray emanating from its center \(O\) intersect at exactly one point.

**Definition 12.4 (Interior and exterior of a circle).** A point \(A\) lies interior of a circle if it lies between the center \(O\) and the point \(P\) where the ray \(\overrightarrow{OA}\) cuts the circle.

A point \(B\) lies exterior of a circle if it lies outside the segment \(OQ\) between the center \(O\) and the point \(Q\) where the ray \(\overrightarrow{OA}\) cuts the circle.

III.3 (Circle-circle intersection property) Let \(\mathcal{C}\) and \(\mathcal{D}\) be two circles. Let \(P, Q\) be two points on circle \(\mathcal{D}\). Assume that point \(P\) lies inside circle \(\mathcal{C}\) and \(Q\) lies outside circle \(\mathcal{C}\). Then the two circles \(\mathcal{C}\) and \(\mathcal{D}\) intersect each other.

III.4 Under the assumptions of (III.3), the two circles intersect in exactly two points, which lie on different sides of the line connecting their centers.
IVa. Axioms of segment congruence

IV.1 If a segment $A'B'$ and a segment $A''B''$ are congruent to the same segment $AB$, then segment $A'B'$ is also congruent to segment $A''B''$.

IV.2 (The sums of congruent segments are congruent.) On a line, let $AB$ and $BC$ be two segments which except for $B$ have no point in common. Furthermore, on the same or another line $a'$, let $A'B'$ and $B'C'$ be two segments which except for $B'$ also have no point in common. In that case,

$$AB \cong A'B' \quad \text{and} \quad BC \cong B'C' \quad \text{then} \quad AC \cong A'C'$$

IV.3 For a circle $C$ with center $O$ and two rays with vertex $O$ which cut the circle at points $P$ and $Q$, the segments $OP$ and $OQ$ are congruent.

IVb. SAS-axiom for triangle congruence

Definition 12.5 (Angle). An angle is the union of two rays with common vertex not lying on one line.

We introduce a segment of unit length.

Definition 12.6 (Congruences of angles). The two angles $\angle POQ$ and $\angle P'O'Q'$ are called congruent if and only if there are four units segments $OP, OQ, O'P', O'Q'$ from the vertices and two congruent segments $PQ$ and $P'Q'$ between the sides of the angles.

IV.4* Two triangles $ABC$ and $A'B'C'$ are congruent if they have a pair of congruent angles and the two pairs of adjacent sides are pairwise congruent. In detail:

The congruences $AB \cong A'B', AC \cong A'C', \angle BAC \cong \angle B'A'C'$ imply the congruences $BC \cong B'C', \angle ABC \cong \angle A'B'C', \angle BCA \cong \angle B'A'C'$

V. Axioms of continuity

V.1 (Axiom of Archimedes) If $AB$ and $CD$ are any segments, then there exists a natural number $n$ such that $n$ segments congruent to $CD$ constructed contiguously from $A$, along a ray from $A$ through $B$, will pass beyond $B$.

V.2 (Cantor’s principle of boxed intervals) Every sequence of boxed intervals contains a common point. In detail:

For a sequence of segments $A_iB_i$ such that

$$A_i \ast A_{i+1} \ast B_{i+1} \ast B_i \quad \text{for all} \quad i = 1, 2, 3, \ldots$$

there exists a point $X^*$ such that

$$A_i \ast X^* \ast B_i \quad \text{for all} \quad i = 1, 2, 3, \ldots$$
VI. Axioms of parallelism

VI.1 (The angle sum of a triangle is two right angles) The sum of the three interior angles of a triangle is two right angles.

VI.2 (Aristotle’s Angle Unboundedness Axiom) For any acute angle $\theta$ and any segment $PQ$, there exists a point $X$ on one side of the angle such that the perpendicular $XY$ dropped onto the other side of the angle is longer than the given segment: $XY > PQ$.

Remark. Based on Proclus commentaries to Euclid, Greenberg has suggested the angle unboundedness axiom. This axiom goes back to Aristotle’s book I of the treatise De Caelo (“On the heavens”).

Remark. As shown in Corollary 28, in every Hilbert plane, the Archimedean Axiom implies Aristotle’s Axiom.

Problem 12.1. Explain, how these axioms imply that each segment is congruent to itself.

Problem 12.2. Explain why congruence of segments is an equivalence relation.

Problem 12.3. Prove that the center of a circle is unique.

Problem 12.4. Prove that every angle is congruent to itself. Explain why congruence of angles is an equivalence relation.

12.2 Fundamental constructions with Euclidean tools

Definition 12.7 (Traditional Euclidean tools). Constructions with traditional Euclidean tools are constructions using only straightedge and compass, done in a finite number of steps of the following types:

1. drawing a line through two different points
2. drawing a circle with a given center through a given point
3. intersecting two lines
4. intersecting a circle and a line
5. intersecting two circles
6. choosing an arbitrary point
7. choosing a point on a given line
8. choosing a point on a given circle
9. deciding whether two points are equal or different

*Remark.* For the enumeration and count of a construction, we do not count producing the given pieces, neither pieces that were only drawn to facilitate the understanding. But producing the resulting pieces is counted.

**Problem 12.5.** Explain how an angle is bisected with traditional Euclidean tools. Enumerate and count the steps needed, according to definition 12.7, and provide a drawing.

*Answer.* Let \( \angle(h, k) \) with vertex \( A \) be given.

1. We choose a point \( B \neq A \) on the ray \( h \).
2. Draw circle(\( A, B \))
3. Find the intersection point \( 2 \cap k \).
4. Draw circle(\( 3, A \))
5. Draw circle(\( B, A \))
6. Find the intersection point \( 4 \cap 5 \) in the interior of the given angle \( \angle(h, k) \).
7. Draw the ray \( \overrightarrow{A, 6} \). This is the angle bisector to be constructed.

We count that seven steps are needed for the construction.

**Problem 12.6.** Given is a line \( l \) and a point lying on the line. Explain how the perpendicular is erected with traditional Euclidean tools.

*Answer.* We draw any circle around the given point \( P \) and let \( A \) and \( B \) be the intersection points of line \( l \) with this circle. Next we draw two circles, with center \( A \) through point \( B \), and with center \( B \) through point \( A \). They intersect in two points, let \( C \) be one of them. The line \( PC \) is the perpendicular to line \( l \) at point \( P \) to be constructed. We get an extra check of accuracy, since this line goes through the other intersection point of the two circles, too.

**Problem 12.7.** Given is a line \( l \) and a point \( P \) not lying on the line. Explain how to construct the reflection point \( P' \) across the line with traditional Euclidean tools, without using line-circle intersection as a means of construction.

*Answer.* We choose any two different points \( A \neq B \) on the given line \( l \) and draw the circles with center \( A \) through point \( B \), and with center \( B \) through point \( A \). They intersect in two points: the given point \( P \) and a point \( P' \neq P \), which is the reflection of point \( P \) across the line \( l \) to be constructed.
Problem 12.8. Given is a line \( l \) and a point not lying on the line. Explain how the perpendicular is dropped with traditional Euclidean tools. Avoid line-circle intersection as a means of construction.

Answer. We choose any two different points \( A \neq B \) on the given line \( l \) and draw the circles with center \( A \) through point \( B \), and with center \( B \) through point \( A \). They intersect in two points: the given point \( P \) and the reflection point \( Q \). The line \( PQ \) is the perpendicular to the line \( l \) through point \( P \) to be constructed.

![Construction of the perpendicular with just two circles and their intersection.](image)

Problem 12.9. Explain that two equilateral triangles the common side of which is any given segment, can be constructed with traditional Euclidean tools. How can the two triangles be distinguished.

Answer. Let the segment \( AB \) be given. We draw the circles with center \( A \) through point \( B \), and with center \( B \) through point \( A \). They intersect in two points \( C \) and \( D \). The triangles \( \triangle ABC \) and \( \triangle ABD \) are the two equilateral triangles with sides \( AB \) to be constructed. They lie on different sides of line \( AB \).

Problem 12.10. Explain how the perpendicular bisector of a segment is constructed with traditional Euclidean tools.

Answer. Let the segment \( AB \) be given. We draw the circles with center \( A \) through point \( B \), and with center \( B \) through point \( A \). They intersect in two points \( C \) and \( D \). The line \( CD \) is the perpendicular bisector segment \( AB \) to be constructed.

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Problem 12.11. Given is a line $l$, a point $A$ not lying on the line, and a point $B$ lying on the line. Construct a circle through points $A$ and $B$ that is tangent to the line. Provide a drawing and explain the major steps.

Answer. We erect the perpendicular at point $B$ onto the given line $l$. Next we construct the perpendicular bisector of segment $AB$. The intersection of these two lines is the center $O$ of the circle through $A$ and $B$, tangent to line $l$, to be constructed.

12.3 Euclidean tools are at least as strong as Hilbert tools

Theorem 12.1 (The traditional Euclidean tools can do at least all constructions possible with Hilbert tools). We assume only the axioms of incidence, order, circles, and congruence as stated above. Under these assumptions, it is also possible to transfer any segment as well as any angle, uniquely in the way postulated by Hilbert’s axioms (III.1) and (III.4). In other words, traditional straightedge and compass are stronger tools than Hilbert tools.

Remark. The axioms of continuity and parallelism are not needed in theorem 12.1. Too, we have shown that Euclid’s collapsible compass can emulate a non-collapsible compass.

Main Theorem 17. The axiom system given above implies Hilbert’s axioms for two-dimensional geometry. On the other hand, the converse is not true. Indeed, it is a more restrictive system.

We now elaborate the steps needed to check Theorem 12.1.
Proposition 12.1 (Transfer of a segment). Given is a segment $AB$ and a ray $a'$ with vertex $A'$. It is always possible to find, with traditional Euclidean tools, a point $B'$ on this ray such that the segment $AB$ is congruent to the segment $A'B'$.

![Figure 12.3: Transfer of a segment with Euclidean tools.](image)

Construction 12.1 (Transfer a segment). Construct an equilateral triangle $\triangle AA'C$, getting point $C$ as an intersection of two circles with center $A$ through point $A'$, and center $A'$ through point $A$.

A segment $AB_1 \cong AB$ on the ray $\overrightarrow{CA}$ can be obtained from the intersection point $B_1$ of this ray with the circle around $A$ through point $B$. A second circle around $C$ through point $B_1$ enable one to get segment $CB_2 \cong CB_1$ on the ray $\overrightarrow{CA'}$, from the intersection point $B_2$ of ray and circle. Finally, the segment $A'B' \cong A'B_2$ on the given ray $a'$ is obtained from the intersection point $B'$ of ray $a'$ with the circle around $A'$ through point $B_2$.

Reason for validity of the construction. Using segment addition or subtraction, it is easy to check that $AB_1 \cong A'B_2$, since by construction both $CB_1 \cong CB_2$ and $CA \cong CA'$. Hence transitivity of congruence implies $AB \cong AB_1 \cong AB_2 \cong AB'$.

Problem 12.12. An alternative construction using more circles, but less lines is shown in the figure on page 327. Describe this construction and give the reason for its validity.
Figure 12.4: Alternative construction to transfer of a segment with Euclidean tools.

**Proposition 12.2 (Transfer of an angle).** Given is an angle $\angle(h,k)$, a ray $h'$ that emanates from any point $A'$, and a half plane of that ray. Then there exists a unique ray $k'$ such that the angle $\angle(h,k)$ is congruent to the angle $\angle(h',k')$ and at the same time all interior points of the angle $\angle(h',k')$ lie on the given side of $h'$.

**Construction 12.2 (Transferring an angle).** We draw a circle around the vertex $A$ of the given angle and get congruent segments $AB \cong AD$ on its sides $h$ and $k$. We transfer segment $AB$ onto the ray $h'$ by means of the construction 12.1 above and get the segment $A'B' \cong AB$.

Finally we transfer segment $BD$ to point $B'$, and get a circle around $B'$ of radius $B'D_2 \cong BD$. This step uses construction 12.1 once more, and hence produces a second equilateral triangle $\triangle BB'E$.

We construct the intersection points $D'$ and $D''$ of the circles around $A'$ of radius $A'B' \cong AB$, and around $B'$ of radius $B'D_2 \cong BD$. One chooses among points $D'$ and $D''$ the one lying in the half plane of $h'$ as required and gets the angle to be constructed: either $\angle B'A'D' \cong \angle BAD$, or $\angle B'A'D'' \cong \angle BAD$.

The intersections of circles and segments need not be postulated, since we can recall:

**Proposition 12.3.** A segment between a point inside a circle and a point outside of this circle intersects the circle in exactly one point.

**Proof.** By Proposition 8.13, the circle-circle intersection property implies the line-circle intersection property. By Proposition 8.10, the line-circle intersection property implies
Figure 12.5: Transfer of an angle with Euclidean tools.

the segment $AB$ between the point $A$ inside and the point $B$ outside of the circle intersects the circle in exactly one point.

We recall some details, for the convenience of the reader. Assume that the circle-circle intersection property holds. Let $l = AB$ be the line between the two given points $A$ inside, and $B$ outside the circle $\mathcal{C}$. The ray $\overrightarrow{OB}$ intersects the circle in a point $Q$ inside the segment $OB$, as postulate by axiom. (IV.2) and given by definition 12.4 of the exterior of a circle.

A special situation occurs in case that the center $O$ of the circle lies on the line $l$. In this case, the intersection point $Q = X$ of the ray $\overrightarrow{OB}$ with the circle is already the required intersection of line and circle. We now exclude that special case and assume that the center $O$ does not lie on line $l$.

The ray $\overrightarrow{OA}$ from the center intersects the circle in a point $P$ outside the segment $OA$ (since $O \neq A$). Recall that reflection images of any point across a line can be
constructed with the intersection of two circles. Let $O'$ and $P', Q'$ be the reflection images of points $O$ and $P, Q$ across the axis $l = AB$. We construct the circle $C'$ with center $O'$ through point $P'$. Points $P'$ and $Q'$ both lie on the reflected circle $C'$. Thus we obtain the figure already shown on page 261. Line $l$ is the perpendicular bisector of

\[ \text{Figure 12.6: A segment from inside to outside a circle intersects the circle—proved from the circle-circle intersection.} \]

the segment $OO'$. Points $O'$ and $P$ lie on the same side of line $l$, since points $O$ and $O'$ lie on different sides of line $l$ but points $O$ and $P$ lie on the same side. Because of proposition 9.1 we know that any point $X$ lies on the same side of line $l$ as point $O$ if and only if $|OX| < |O'X|$.

Because points $P$ and $O'$ lie on the same side of line $l$, we get $|OP'| = |O'P| < |OP|$. Hence point $P'$ lies inside circle $C$. Because points $O$ and $O'$ lie on different sides of line $l$ and points $O$ and $Q$ lie on the same side, the points $Q$ and $O'$ lie on different sides of line $l$ and $|OQ| < |O'Q| = |OQ'|$. Hence point $Q'$ lies outside circle $C$.

Because point $P'$ lies inside, but point $Q'$ lies outside the circle $C$, and both points lie on the reflected circle $C'$, the circle-circle intersection property implies that the two circles intersect in some point $X$. Since it has the same distance from both centers $O$ and $O'$, by proposition 9.1 point $X$ lies on the axis $l$. Thus we have obtained an intersection point of the given circle $C$ and line $AB$. 

Remark. In this derivation of the line-circle intersection property, we had to distinguish the cases that the given line goes through the center or not $O$. This is an instance were
the item "deciding whether two points are equal or different" is actually needed.

In the special case of a line through the center, we need to invoke the circle axiom (IV.2) directly, since the reflection across the line does not produce a different second circle. In case of a line passing closely to the center, it is still very impractical to use circle-circle intersection because of the small angle at the intersection of the two circles, and the direct application of the circle-line intersection leads to a more accurate construction. Because of that, the line-circle intersection is a practical tool useful to improve the accuracy of constructions.

Too, we have shown in Proposition 8.11:

**Proposition 12.4.** If a circle has points on both sides of a line, then it intersects the line in two points.

**Problem 12.13.** Given is a line $l$ and a point not lying on the line. Explain how the perpendicular is dropped using a circle-line intersection.

**Problem 12.14.** The figure on page 12.3 shows a variant construction for the transfer of an angle. Enumerate and count the number of elementary steps according to definition 12.7 needed to perform the construction. Which additional assumption is made in this construction.

**Solution.** 1. Draw circle$(A, B)$. 
2. Draw circle($B, A$).

3. Find an intersection points $X$ of $1 \cap 2$.

4. Find the second intersection point $Y$ of $1 \cap 2$.

5. Draw the line $p = XY$. This is the perpendicular bisector of segment $AB$.

6. Find the intersection points $C = h \cap p$.

7. Draw circle($A, C$).

8. Find the intersection $D = k \cap 7$.

9. Draw ray $\overrightarrow{CB}$.

10. Draw circle($C, D$).

11. Find the intersection point $F = 9 \cap 10$.

12. Draw circle($B, F$).

13. Find the intersection point $G = l \cap 12$.

14. Draw circle($B, C$).

15. Find the intersection point $E = l \cap 14$.

16. Draw circle($E, G$).

17. Find an intersection point of $14 \cap 16$.

18. If this point does not lie in the half plane as required, find the second intersection point $K$ of $14 \cap 16$.

19. Draw the ray $\overrightarrow{BK}$.

   The angle $\angle EBK$ is the transfer of the given angle $\angle(h, k)$ onto the given ray $l = \overrightarrow{BE}$ to be constructed. We see that 18 or 19 steps are needed—provided that intersection point $C$ does exist.

**Remark.** We see that the construction made the additional assumption that one side of the given angle intersects the perpendicular bisector of the vertices of the given angle and the given ray. In Euclidean geometry this assumption is not too restrictive, because it always holds for one side of either the given angle or its vertical angle. In hyperbolic geometry, the restriction is more severe.
Figure 12.8: Transfer 2 of an angle.
Problem 12.15. Consider transfer 2 of an angle shown in the figure on page 12.3. Enumerate and count the number of elementary steps according to definition 12.7 needed to perform the construction.

Solution.

1. Draw circle($A,B$).
2. Draw circle($B,A$).
3. Find an intersection points $F$ of $1 \cap 2$.
4. Find the second intersection point $G$ of $1 \cap 2$.
5. Draw the line $FG$.
6. Draw line $AB$.
7. Find the intersection point $M = 5 \cap 6$.
8. Find the intersection point $C = h \cap 1$.
9. Find the intersection point $D = k \cap 1$.
10. Draw ray $\overrightarrow{CM}$.
11. Find the intersection point $C' = 2 \cap 10$.
12. Draw ray $\overrightarrow{DM}$.
13. Find the intersection point $D' = 2 \cap 12$.
14. Draw circle($D',B$).
15. Find the intersection point $E = l \cap 2$.
16. Draw circle($E,B$).
17. Find the intersection point $K = 14 \cap 16$.
18. Draw circle($K,C'$).
19. Find an intersection point of $H = 2 \cap 18$.
20. If this point $H$ does not lie in the half plane as required, draw circle($E,H$).
21. Find the new intersection point $H := 2 \cap 20$.
22. Draw ray $\overrightarrow{BH}$.

The angle $\angle EBH$ is the transfer of the given angle $\angle(h,k)$ onto the given ray $l = \overrightarrow{BE}$ to be constructed. We see that 20 or 22 steps are needed. This construction does work always.