2 Homework

Theorem 1. The four-momentum of a particle moving in any field of forces is

\[ p^\mu = \left[ E/c, \vec{p} \right] \]

The total energy \( E \) and the rest mass \( m \) are related by

(2.1) \[ E = \sqrt{m^2c^4 + c^2\vec{p}^2} \]

The momentum and the velocity by

(2.2) \[ \vec{p} = \frac{E}{c^2} \vec{v} \]

These formulas retains their meaning for \( m = 0 \), as does occur for a photon, or other massless particle.

Problem 2.1. A particle is non-relativistic if \( c|\vec{p}| \ll mc^2 \), or equivalently \( \beta \ll 1 \). Use the power expansion

\[ \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} \pm \ldots \]

to get the approximation of a kinetic energy for a non-relativistic particle.

Problem 2.2. Check that the kinetic energy of the scattered electron is

\[ T' = mc^2 \frac{\lambda_0^2}{\lambda \lambda'} (1 - \cos \theta) \]

Take as an example the monochromatic Mo-K\(_\alpha\)-rays from A.H. Compton’s experiment of 1922. The incident wavelength is \( \lambda = 70 \text{ pm} \). Calculate the kinetic energy of the scattered electron.

Problem 2.3. Determine the direction of the scattered electron, from the \( y \)-component of the four-momentum conservation.

(a) Write down the \( y \)-component of the four-momentum conservation.
(b) Get $\sin \phi$ in terms of $\sin \theta$, and the wave length $\lambda'$ and relativity parameter $\gamma'$ for the scattered electron.

(c) From the kinetic energy $T'$ obtained in the previous problem see that

$$\gamma' - 1 = \frac{\lambda_C^2}{\lambda} (1 - \cos \theta)$$

Use this expression to simply. After a bid lengthy calculation one gets $\sin \phi = -A \cos(\theta/2)$ with a factor $A < 1$ which is approaching one for non-relativistic electron.

(d) Express the result $\sin \phi = -\cos(\theta/2)$, to be expected in the experiments, in simple geometric terms.

**Problem 2.4.** Rotate the lab-system by the angle $\phi$ such that the scattered electron moves along the positive x-axis. Transform the components of the four-momenta for the incoming and scattered photon and electron to this system. Convince yourself that

$$\frac{\sin(\phi - \theta)}{\sin \phi} = \frac{\omega}{\omega'} = 1 + \frac{\lambda_C}{\lambda} (1 - \cos \theta)$$

Conclude that for an experiment with $\lambda \gg \lambda_C$, one gets indeed $\phi \approx \theta/2 - 90^\circ$.

**Problem 2.5.** Inverse Compton scattering\(^1\) occurs whenever a photon scatters off a particle moving with almost speed of light. Suppose that a particle with rest mass $M$ and total energy $E$ collides head on with a photon of energy $E_\gamma$. For simplicity, assume that the scattered particles move back along the same axis.

(i) Convince yourself that the conservation of the four-momentum implies

$$E_\gamma (E + cp) = E'_\gamma (E - cp) + 2E_\gamma E'_\gamma$$

(ii) Solve for $E'_\gamma$ and use that $Mc^2 \ll E$ and hence $E + cp \approx 2E$ holds with any accuracy needed. Show that under this assumption the scattered photon has the energy

$$E'_\gamma = \frac{4E^2 \cdot E_\gamma}{M^2c^4 + 4E \cdot E_\gamma}$$

(iii) Such a situation may occur for example for a collision of an ultra-relativistic cosmic ray with a photon from the cosmic background radiation. How much energy can such a cosmic ray proton transfer to a microwave background photon?

Energies up to $E = 10^{20}$ eV may occur in cosmic rays. The proton has rest energy $Mc^2 = 938.272$ MeV. A typical energy of photon from the cosmic background radiation is $E_\gamma = 2.7^\circ K \cdot 8.67 \cdot 10^{-5}$ eV/K.

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\(^1\)See Hobson *General Relativity* p.132 problem 5.14
Definition 1 (Center of mass system). For any set of colliding particles, the center of mass system or CMS, is the inertial system for which the total momentum of the incoming particles is zero. Thus their four-momentum is \([Mc, 0, 0, 0]\) where \(M\) is the total rest mass.

Problem 2.6. Consider the Compton scattering process in the CMS system, with incoming photon moving in the positive x-direction. Write down the four-momenta of the incident photon and electron, and the possible four-momenta for the scattered photon and electron.

Problem 2.7. The Bevatron at Berkeley was built with the idea of producing antiprotons, \(^2\) by the reaction \(p + p \rightarrow p + p + p + p\). Thus one intended to let a high-energy proton strike a proton at rest;—at that point of history, it was not yet possible to send two proton rays against each other. By known conservation laws, it was clear that only an additional pair of proton and antiproton could be expected. Find the threshold for the energy \(E\), and kinetic energy \(T\), of the incoming proton at which this reaction becomes possible.

(i) Calculate the total energy-momentum four-vector \(p_{\text{tot}}\) of the incoming particles in the lab system.

(ii) At the threshold, the created four particles cannot have any additional kinetic energy. Thus they need to be at rest in the CMS-system. Calculate the total energy-momentum four-vector \(p'_{\text{tot}}\) of the scattered particles in the CMS-system.

(iii) From the conservation of the four-momentum and Lorentz invariance, we know that

\[ p_{\text{tot}} \cdot p_{\text{tot}} = p'_{\text{tot}} \cdot p'_{\text{tot}} \]

(iv) Determine \(E\) and \(|p|\) from the two equations

\[ (E + Mc^2)^2 - c^2(p^2) = (4Mc^2)^2 \]
\[ E^2 - c^2(p^2) = M^2c^4 \]

(v) Determine the kinetic energy \(T\) and the speed of the incoming protons at threshold.

Problem 2.8 (Twin paradox or travelling keeps young). For this problem, I put \(c = 1\) and consider only one space dimension. We do a series of simplifying calculations, and prove the conjecture at first in \(1 + 1\) dimensions. Confirm for any two time-like vectors \(x = (t, x) \in \text{Future}\) and \(p = (E, p) \in \text{Future}\), the reversed triangle inequality

\[ |x + p| \geq |x| + |p| \]

holds—even with proper inequality \(>\) unless they are proportional.

\(^2\)See also D. Griffiths Introduction to Elementary Particles, p.106