Hyperbolic Geometry and the Pseudo-sphere

A Senior Project, based on Math 3181

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Introduction.

Through the work of Gauss on differential geometry, it became clear—after a painfully slow historic process—that there is a model of hyperbolic geometry on surfaces of constant negative Gaussian curvature. One particularly simple such surface is the pseudosphere.

Due to the ideas forwarded by Gauss, mathematicians have in the end advanced to the concept of a curved surface as a space of its own interest. Gauss' work implied that there are non-Euclidean geometries at least on surfaces regarded as spaces in themselves. Whether Gauss himself already saw this non-Euclidean interpretation of his geometry of surfaces is not clear. (Morris Kline III, p.888). Continuing Gauss work, Riemann and Minding have thought about surfaces of constant negative curvature, but neither man related these to hyperbolic geometry. Independent of Riemann, Beltrami recognized that surfaces of constant curvature are non-Euclidean spaces. As we explain in detail below, Beltrami shows that on a rotation surface of negative constant curvature, which is called a pseudo-sphere, one can realize a *piece* of the hyperbolic plane. An obvious important idea is finally spelled out!

But the idea comes with a disappointment: by a result of Hilbert, there is no regular analytic surface of constant negative curvature on which the geometry of the *entire* hyperbolic plane is valid. The final outcome is a trade off between the pseudo-sphere and the Poincaré disk. As models for hyperbolic geometry, both have their strengths and weaknesses.

The pseudo-sphere is a model for a *limited portion* of the hyperbolic plane. Angles are represented correctly, and the arc length of a geodesic is the *correct hyperbolic distance*. Furthermore, because of the constant Gaussian curvature, on the pseudo-sphere a figure may be shifted about and just bending will make it conform to the surface. The situation is similar to the more familiar case of Euclidean geometry on a circular cylinder or cone. As everybody knows, on a circular cylinder, a plane figure can be fitted by simply bending it, without stretching and shrinking.

On the other hand, the Poincaré disk is a model for the *entire* hyperbolic plane. Angles are represented correctly, but the price one finally has to pay is that *hyperbolic distances are distorted*. The hyperbolic lines become circular arcs, perpendicular to the ideal boundary. One can see the distortion easily in Esher's superb artwork, which uses tilings of the hyperbolic plane with congruent figures.

This trade off just explained makes the isometry between the pseudo-sphere into the Poincaré disk especially interesting. One such isometric mapping is explicitly constructed below. Hilbert's result gets rather natural, too. As explained below, in the sense of hyperbolic geometry, the boundary of the pseudo-sphere turns out to be a horocyle.

About Gauss' differential geometry.

Karl Friedrich Gauss had devoted an immense amount of work to geodesy and map making, starting 1816. This stimulus leads to his definitive paper in differential geometry of 1827: "Disquisitiones Generales circa Surperficies Curvas" . In this work, Gauss introduces the basics of curved surfaces, and goes far beyond. One can imagine curved surfaces as simply embedded in three dimensional space. Though, due to the ideas forwarded by Gauss, mathematicians have in the end advanced to the concept of a curved surface as a space of its own interest.

The distance ds of neighboring points on the surface with parameters (u, v) and $(u + du, v + dv)$ is given by the first fundamental form

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

As shown by formula (6.3) below, the first fundamental form can be calculated easily, once an embedding of the surface into three dimensional space $\mathbb{R}^3$ is given by some parametric equations $x = x(u, v)\,,\ y = y(u, v)\,,\ z = z(u, v)$. The geodesics on curved surfaces are defined to be the shortest curves lying on the given surface, connecting any two given points. Gauss work sets up the differential equation for the geodesics. Gauss introduces the two main curvatures, called κ_1, κ_2. They turn out to be simply the extremal curvatures of normal sections of the surface. A new important feature is the Gaussian curvature, called K. Gauss shows that $K = \frac{LN-M^2}{EG-F^2}$, the quotient of the determinants of the second and first fundamental form. But, even simpler, the Gaussian curvature turns out to be the product of the two principle curvatures:

$$K = \kappa_1\kappa_2 \tag{1}$$

Gauss shows the remarkable fact that this curvature is preserved during the process of bending the curved surface inside a higher dimensional space, without stretching, contracting or tearing it. On the contrary, the two main curvatures are changed by flexing the surface.

There are actually at least two different proofs for this fact contained in Gauss' work. The first one depends on Gauss' characteristic equation

$$K = \frac{1}{2H}\frac{\partial}{\partial u}\left[\frac{F}{EH}\frac{\partial E}{\partial v} - \frac{1}{H}\frac{\partial G}{\partial u}\right] + \frac{1}{2H}\frac{\partial}{\partial v}\left[\frac{2}{H}\frac{\partial F}{\partial u} - \frac{1}{H}\frac{\partial E}{\partial v} - \frac{F}{EH}\frac{\partial E}{\partial u}\right] \tag{2}$$

where $H = \sqrt{EG - F^2}$. Obviously, any such equation implies that the Gaussian curvature depends only on the first fundamental form. The first fundamental form is preserved, if one bends the curved surface in three space, without stretching, contracting or tearing it. Therefore the functions E, F, G, H which determine the first fundamental form depend only on the parameters (u, v), but do not depend at all on how—or even whether at all—the surface lies in three space. Because of the Gauss' characteristic equation (2), the same is true for the Gaussian curvature K. Because of all that, one says that the Gaussian curvature is an intrinsic property of the curved surface.

1. | | 10 | Use the fundamental form for the Poincaré disk model

$$(ds_D)^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} \tag{3}$$

to calculate its Gaussian curvature.

Solution. Formula (3) implies that the functions in the first fundamental form are $E = G = H = 4(1 - x^2 - y^2)^{-2}$ and $F = 0$. Hence, with $x = u$ and $v = y$, we get from formula (2)

$$\begin{aligned}
K &= \frac{1}{2E}\frac{\partial}{\partial x}\left[-\frac{1}{E}\frac{\partial E}{\partial x}\right] + \frac{1}{2E}\frac{\partial}{\partial y}\left[-\frac{1}{E}\frac{\partial E}{\partial y}\right] = -\frac{1}{2E}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\ln E \\
&= +\frac{1}{E}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\ln(1 - x^2 - y^2) = +\frac{1}{E}\left(\frac{\partial}{\partial x}\frac{-2x}{(1 - x^2 - y^2)} + \frac{\partial}{\partial y}\frac{-2y}{(1 - x^2 - y^2)}\right) \\
&= \frac{-2}{E}\left(\frac{1 - x^2 - y^2 + 2x^2}{(1 - x^2 - y^2)^2} + \frac{1 - x^2 - y^2 + 2y^2}{(1 - x^2 - y^2)^2}\right) = \frac{(-2)\cdot 2}{4} = -1
\end{aligned}$$

By the way, the result $K = -1$ motivates the annoying factor 4 in formula (3). □

The most enlightened proof that the Gaussian curvature is an intrinsic property of the surface (I would say) uses Gauss' notion of integral curvature. For any domain G on a given curved surface, the integral curvature is defined as the integral $\int\int_G KdA$, where dA denotes the area element of the surface.

Take a geodesic triangle T. This is the region bounded by the geodesics between any three given points A, B, C on the surface. Let α, β, γ be the angles (between tangents) to the geodesics at the three vertices. Angles are measured in radians, in this context. Gauss proves

$$\int\int_T KdA = \alpha + \beta + \gamma - \pi \tag{4}$$

The quantity on the right hand side is the deviation of the angle sum $\alpha+\beta+\gamma$ from the Euclidean value $180°$. Of course, in radian measures, the Euclidean angle sum is π. The quantity $\alpha+\beta+\gamma-\pi$ is called the excess of the triangle $\triangle ABC$. For the hyperbolic case, the excess is negative. In that case, one calculates using the excess times -1, which is called defect. In words, Gauss' theorem tells the following:

For a geodesic triangle, the integral curvature equals the excess of its angle sum.

This theorem, Gauss says, ought to be counted as a most elegant theorem. Indeed, instead of the complicated characteristic equation (2), one has a simple property of a geodesic triangle from which to derive the Gaussian curvature.

I discuss a few immediate, but important consequences of (4). First of all, as an immediate implications of (4), the Gaussian curvature is an intrinsic property of a curved surface. Recall that both geodesics, as well as measurement of area depend only on the first fundamental form. Hence, because of (4), the same is true for the Gaussian curvature.

Another easy consequence of (4) is obtained from the special case of a sphere. For this surface, the Gaussian curvature is constant, and equal to $K = \dfrac{1}{R^2}$ where R is the radius of the sphere. Hence one obtains for the area of a spherical triangle $A = (\alpha + \beta + \gamma - \pi)R^2$. as was already known before Gauss, e.g. to Lambert.

2. | | 10 | Tile a sphere by equilateral triangles. It can be done in three ways:

(i) Four triangles with $\alpha = \beta = \gamma = 120°$.
(ii) Eight triangles with $\alpha = \beta = \gamma = 90°$.
(iii) N triangles with $\alpha = \beta = \gamma = 72°$.

Explain and draw these tilings. To which Platonic bodies do the vertices correspond? Determine the surface area of the sphere from (i) and (ii), then get the number N in (iii).

Solution. From item (i), the area of the sphere is

$$A = 4 \cdot \left(3\frac{2\pi}{3} - \pi\right) R^2 = 4\pi R^2$$

The vertices of the four triangles form a tetrahedron. Similarly, item (ii) yields

$$A = 8 \cdot \left(3\frac{\pi}{2} - \pi\right) R^2 = 4\pi R^2$$

The vertices of the eight triangles form a octahedron.
We can now calculate the number N of triangles in the tiling (iii). Because of

$$A = N \cdot \left(3\frac{2\pi}{5} - \pi\right) = 4\pi R^2$$

one gets $N = 20$. The vertices of the twenty triangles form an icosahedron. □

3. | | 10 | Here is a further important consequence of (4).

A common bound for the area of all triangles. *On a surface with negative constant Gaussian curvature $K < 0$, the area of any triangle is less than $\frac{\pi}{-K}$.*

On December 17, 1799, Gauss wrote to his friend, the Hungarian mathematician Wolfgang Farkas Bolai (1775-1856):

* *As for me, I have already made some progress in my work. However, the path I have chosen does not lead at all to the goal which we seek [deduction of the parallel axiom], and which you assure me you have reached. It seems rather to compel me to doubt the truth of geometry itself. It is true that I have come upon much which by most people would be held to constitute a proof; but in my eyes it proves as good as nothing. For example, if we could show that a rectilinear triangle whose area would be greater than any given area is possible, then I would be ready to prove the whole of Euclidean geometry absolutely rigorously.*
* *Most people would certainly let this stand as an axiom; but I, no! It would indeed be possible that the area might always remain below a certain limit, however far apart the three angular points of the triangle were taken.*

From about 1813 on Gauss developed his new geometry. He became convinced that it was logically consistent and rather sure that it might be applicable. His letter written in 1817 to Olbers says:

* *I am becoming more and more convinced that the physical necessity of our Euclidean geometry cannot be proved, at least not by human reason nor for human reason. Perhaps in another life we will be able to obtain insight into the nature of space, which is now unattainable. Until then we must place geometry not in the same class with arithmetic, which is purely a priori, but with mechanics.*

(a) Find two further enlightening statements of Gauss, and comments on all four statements.
(i) Are they courageous?
(ii) Are they to the benefit of the scientific community?
(iii) Are they helpful for the person he addresses?
(iv) Are they just against other people?
(v) What would you have done in Gauss' place?
(b) Choose two of Gauss' comments. Write a letter as you imagine you would have written in place of Gauss.

4. | | 10 | To test the applicability of Euclidean geometry and his non-Euclidean geometry, Gauss actually measured the sum of the angles of the triangle formed by three mountain peaks in middle Germany: Broken, Hohehagen, and Inselberg. the sides of the triangle were $69, 85, 197$ km. His measurement yielded that the angle sum exceeded 180° by 14".85.

(a) Use Herons formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ to calculate the area of the triangle.

(b) Take $R = 6378$ km as radius of the earth. Calculate the angle excess for a spherical triangle between the three mountain peaks. You need to convert angular measurements!

1 radian equals $\frac{180^\circ}{\pi} = \frac{180 \cdot 3600''}{\pi}$.

(c) Is the triangle that Gauss measured actually a spherical triangle.
Why or why not?

(d) Reflect on the motives why Gauss did his measurement. Find and read some further sources. Think of the following and further motives and possibilities. Did Gauss really just want to

(i) check accuracy?
(ii) check geometry?
(iii) It was just a theoretical thought experiment, not really performed.

5. | | 10 | Before introducing the pseudo-sphere, I do some investigation about more general rotation surfaces. We take the graph of an arbitrary function $y = f(x)$, and rotate it about the y-axis to produce a rotation surface in three dimensional space.

Principal curvatures and Gaussian curvature for rotation surfaces. *The first principle curvature of a rotation surface in the xy plane is*

$$\kappa_1 = \frac{y''}{(1+y'^2)^{\frac{3}{2}}} \tag{5.1}$$

This is just the curvature of the graph $y = f(x)$.

Recall that the perpendicular to the tangent of a curve is called the normal of the curve. The second principal curvature occurs for a section of the surface by a plane $\mathcal{P}_2$, *which intersects the xy plane along the normal of the curve* $y = f(x)$, *and is perpendicular to the xy plane. The second principal curvature is*

$$\kappa_2 = \frac{y'}{x(1+y'^2)^{\frac{1}{2}}} \tag{5.2}$$

The Gaussian curvature of the rotation surface is the product

$$K = \frac{y'\,y''}{x(1+y'^2)^2} \tag{5.3}$$

Here is an argument to justify (5.2): Let $\tan\beta = y'$ be tangent of the slope angle for $y = f(x)$, as usual. Calculate $\sin\beta$. Calculate the hypothenuse AB of the right $\triangle ABC$, with vertex $A = (x, y)$ on the curve, leg AC parallel to the x-axis, and hypothenuse AB perpendicular to the curve. One can show that point B is the center of the best approximating circle in the plane $\mathcal{P}_2$. Hence $\kappa_2 = \frac{1}{AB}$. Use this idea to get the main curvature κ_2.

Answer. $\tan\beta = \frac{\overline{AC}}{\overline{BC}} = y'$ Hence

$$\sin\beta = \frac{\overline{AC}}{\overline{AB}} = \frac{\overline{AC}}{\sqrt{\overline{BC}^2 + \overline{AC}^2}} = \frac{y'}{\sqrt{1+y'^2}}$$

$$\kappa_2 = \frac{1}{\overline{AB}} = \frac{\sin\beta}{x} = \frac{y'}{x(1+y'^2)^{\frac{1}{2}}}$$

□

6. | | 10 | Here is another way to obtain the Gaussian curvature (5.3). On the rotation surface, I choose as parameter $u = \phi$ the rotation angle, and as parameter $v = r$ the distance from the rotation axis, which I take to be the y-axis, as above. Hence the surface of rotation has the parametric representation

$$\begin{aligned} x &= v\cos u \\ y &= f(v) \\ z &= v\sin u \end{aligned} \tag{6.1}$$

The derivatives by the parameters are

$$\begin{aligned} x_u &= -v\sin u & x_v &= \cos u \\ y_u &= 0 & y_v &= f'(v) \\ z_u &= v\cos u & z_v &= \sin u \end{aligned} \tag{6.2}$$

From these derivatives, one gets the first fundamental form, by the general formulas

$$\begin{aligned} E &= x_u^2 + y_u^2 + z_u^2 \\ F &= x_u x_v + y_u y_v + z_u z_v \\ G &= x_v^2 + y_v^2 + z_v^2 \end{aligned} \tag{6.3}$$

Now calculate E, F, G and the first fundamental form from (6.2) and (6.3), and get the root of the determinant $H = \sqrt{EG - F^2}$. Finally calculate the Gaussian curvature from the characteristic equation (2).

Solution. One gets $E = v^2$, $F = 0$ and $G = 1 + f'^2$ and

$$ds^2 = v^2 du^2 + (1 + f'^2) dv^2 = r^2 d\phi^2 + (1 + f'(r)^2) dr^2$$

The root of the determinant is $H = v\sqrt{1 + f'^2}$. Because all four quantities E, F, G, H depend only on v, the partial derivatives by u all vanish. Thus Gauss' characteristic equation (2) can be simplified to yield

$$\begin{aligned} K &= \frac{1}{2H} \frac{\partial}{\partial u} \left[\frac{F}{EH} \frac{\partial E}{\partial v} - \frac{1}{H} \frac{\partial G}{\partial u} \right] + \frac{1}{2H} \frac{\partial}{\partial v} \left[\frac{2}{H} \frac{\partial F}{\partial u} - \frac{1}{H} \frac{\partial E}{\partial v} - \frac{F}{EH} \frac{\partial E}{\partial u} \right] \\ &= \frac{1}{2H} \frac{\partial}{\partial v} \left[-\frac{1}{H} \frac{\partial E}{\partial v} \right] = -\frac{1}{2H} \frac{d}{dv} \left[\frac{2v}{H} \right] \end{aligned}$$

Now use $H = v(1 + f'^2)^{\frac{1}{2}}$ and go on. We arrive once more at formula (5.3):

$$\begin{aligned} K &= -\frac{1}{2H} \frac{d}{dv} \left[\frac{2v}{H} \right] = -\frac{1}{v\sqrt{1 + f'^2}} \frac{d(1 + f'^2)^{-\frac{1}{2}}}{dv} \\ &= \frac{1}{2v\sqrt{1 + f'^2}} \left[(1 + f'^2)^{-\frac{3}{2}} \right] 2f'f'' = \frac{f'f''}{v(1 + f'^2)^2} \end{aligned}$$

7. | | 10 | Calculate the Gaussian curvature of a three dimensional sphere of radius a.

Solution. The sphere is provided by rotating the graph of $x^2 + y^2 = a^2$ about the y-axis. Implicit differentiation yields $2x + 2yy' = 0$ and hence

$$\begin{aligned} y' &= -\frac{x}{y} \\ y'' &= -\frac{1 \cdot y - xy'}{y^2} = \frac{xy' - y}{y^2} = \frac{-x^2/y - y}{y^2} = -\frac{a^2}{y^3} \\ K &= \frac{y'y''}{x(1 + y'^2)^2} = \frac{a^2}{y^4 \left(1 + x^2/y^2\right)^2} = \frac{a^2}{(x^2 + y^2)^2} = \frac{1}{a^2} \end{aligned}$$

□

8. | | 10 | The issue is now to find a rotation surface of constant <u>negative</u> Gaussian curvature $K = -a^{-2}$. Such a surface is called <u>pseudo-sphere</u>. Use the formula (5.3) for the Gaussian curvature, and get a differential equation of first order for the function $u := y'^2$. You may begin by getting the derivative $\frac{du}{dx}$.

Solution. The derivative of the function $u := y'^2$ is $\frac{du}{dx} = 2y'y''$. Next, I put the requirement $K = -a^{-2}$ into the formula (5.3). One gets

$$\begin{aligned} \frac{y'\,y''}{x(1+y'^2)^2} &= -\frac{1}{a^2} \\ 2y'\,y'' &= -\frac{2x(1+y'^2)^2}{a^2} \\ \frac{du}{dx} &= -\frac{2x(1+u)^2}{a^2} \qquad (8) \end{aligned}$$

□

9. | | 10 | Solve the differential equation

$$\frac{du}{dx} = -\frac{2x(1+u)^2}{a^2} \qquad (8)$$

by separation of variable. For simplicity use the initial data $u(a) = 0$, to get a curve through the point $x = a, u = 0$.

Solution.

$$\begin{aligned} \frac{du}{dx} &= -\frac{2x(1+u)^2}{a^2} \\ \frac{du}{(1+u)^2} &= -\frac{2xdx}{a^2} \\ \int_a^u \frac{du}{(1+u)^2}\,du &= -\int_0^x \frac{2xdx}{a^2}\,dx \\ \left[-\frac{1}{1+u}\right]_a^u &= \left[-\frac{x^2}{a^2}\right]_0^x \\ -\frac{1}{1+u} + 1 &= -\frac{x^2}{a^2} + 1 \\ 1+u &= \frac{a^2}{x^2} \\ u &= \frac{a^2-x^2}{x^2} \end{aligned}$$

□

10. | | 10 | Check that the differential equation

$$y' = -\frac{\sqrt{a^2 - x^2}}{x}$$

with $|x| \leq a$, has the solution

$$y = a \ln \left[\frac{a + \sqrt{a^2 - x^2}}{x} \right] - \sqrt{a^2 - x^2} + C \tag{10}$$

Find the general solution of the equation

$$y' = +\frac{\sqrt{a^2 - x^2}}{x}$$

Answer.

$$y = a \ln \left[\frac{a - \sqrt{a^2 - x^2}}{x} \right] + \sqrt{a^2 - x^2} + C$$

□

11. | | 10 | The rotation surface with

$$y = a \ln \left[\frac{a + \sqrt{a^2 - x^2 - z^2}}{\sqrt{x^2 + z^2}} \right] - \sqrt{a^2 - x^2 - z^2}$$

is called the <u>pseudo-sphere</u>. I have chose the y-axis as axis of rotation. Check, once more, that the Gaussian curvature of the pseudo-sphere is $K = -a^{-2}$. You may refer to the calculations above.

Solution. □

11a. | | 10 | Check the following fact: The segment on the tangent to the curve (10), between the touching point A, and the intersection B of the tangent with the y-axis has always the same length a. For that reason, the curve (10) is called <u>tractrix</u>.

Reason. Take the right $\triangle ABC$, form by the tangent at point A, and the perpendicular from A onto the y-axis. (This triangle is different from the one in item 5.) We know that

$$y' = \tan \alpha = \frac{\overline{BC}}{\overline{AC}}$$

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 = x^2(1 + y'^2) = a^2$$

□

12. | | 10 | The surface area of of rotation surface, made by rotating $y = f(x)$ about the y-axis, for $x_1 \leq x \leq x_2$ is

$$S = 2\pi \int_{x_1}^{x_2} x\sqrt{1 + {y'}^2}\, dx$$

Calculate the surface of the pseudo-sphere for bounds $0 < x \leq a$.

Answer. Because of $1 + {y'}^2 = 1 + u = \frac{a^2}{x^2}$, we get

$$S = 2\pi \int_0^a x\frac{a}{x}\, dx = 2\pi a^2$$

□

13. | | 10 | I introduce now two convenient coordinates on the pseudo-sphere. As first coordinate, I choose the angle of rotation ϕ about the y-axis. A second convenient coordinate is the radius $r = \sqrt{x^2 + z^2}$ measured from the axis of rotation, Up to now it was called x, but now I choose to name it r. The three parameters r, ϕ, y are cylindrical coordinates of three dimensional space. The first two of them are convenient parameters on the pseudo-sphere.

Riemann Metric for the Pseudo Sphere. *The infinitesimal distance ds of points with coordinates* (ϕ, r) *and* $(\phi + d\phi, r + dr)$ *is*

$$ds^2 = r^2 d\phi^2 + \frac{a^2}{r^2} dr^2 \tag{13}$$

Reason. The distance on the pseudo-sphere is calculated from the usual Euclidean distance for points of the three dimensional space into which the surface is embedded. At first, I convert the distance from Cartesian to cylindrical coordinates. Because the y-axis is the rotation axis, its coordinate stays, but the pair (x, z) is converted to polar coordinates. Hence one gets

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\phi^2 + dy^2 \tag{13.1}$$

Now we restrict to points on the pseudo- sphere. Hence the coordinates r and y are related in the same way as x and y before. Thus $y' = -\frac{\sqrt{a^2-x^2}}{x}$ gets

$$\frac{dy}{dr} = -\frac{\sqrt{a^2 - r^2}}{r} \tag{13.2}$$

Now we use (12.2) to eliminate y from (12.1) and get

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\phi^2 + dy^2 = dr^2 + r^2 d\phi^2 + \left(\frac{dy}{dr}\right)^2 dr^2 \\ &= dr^2 + r^2 d\phi^2 + \frac{a^2 - r^2}{r^2} dr^2 = r^2 d\phi^2 + \frac{a^2}{r^2} dr^2 \end{aligned}$$

as to be shown. As an alternative, we can the first fundamental form calculated in item 6. Since $1 + f'(r)^2 = 1 + \frac{a^2 - r^2}{r^2} = \frac{a^2}{r^2}$, one gets again

$$ds^2 = r^2 d\phi^2 + (1 + f'(r)^2) dr^2 = r^2 d\phi^2 + \frac{a^2}{r^2} dr^2$$

□

14. | | 10 | The goal is now to construct an isometric mapping from the pseudo-sphere to the Poincaré disk. It is convenient to get this mapping as composition of a mapping from the pseudo-sphere to the halfplane, and the conformal mapping from the halfplane to the disk.

Throughout, we denote the upper open halfplane by $H = \{(u, v) : v > 0\}$. Its boundary is just the real axis $\partial H = \{(u, v) : v = 0\}$.

The Isometric Mapping between the Half Plane and the Disk. *The linear fractional function*

$$z = \frac{iw + 1}{w + i} \tag{14}$$

is a conformal mapping and a bijection from $\mathbf{C} \cup \{\infty\}$ *to* $\mathbf{C} \cup \{\infty\}$*. Especially, one easily checks that*

$$w = \infty \mapsto z = i\ , \quad w = i \mapsto z = 0\ , \quad w = -i \mapsto z = \infty$$

The inverse mapping is

$$w = \frac{1 - iz}{z - i} \tag{14.inv}$$

The mapping (14) *preserves angles, the cross ratio and the hyperbolic distance, and maps generalized circles to generalized circles.*

The upper half plane $H = \{w = u + iv : v > 0\}$ *is mapped by* (14) *onto the disk* $D = \{z = x + iy : x^2 + y^2 < 1\}$.

The Poincaré half plane model of hyperbolic geometry is constructed from the Poincaré disk model via the isometry (14). One translates the definitions for the disk model to the halfplane model.

The points of H are the `"points"` for Poincaré's model. The points of ∂H are called `"ideal points"` or `"endpoints"`. Those are not points for the hyperbolic geometry.

The inversion image of any point $P = (u, v)$ mapped via reflection by the real axis is $P' = (u, -v)$. It is convenient to use complex numbers. In complex notation, reflection by the real axis is complex conjugation: point $P = w = u + iv$ is reflected to $P' = \overline{w} = u - iv$.

The `"lines"` for Poincaré's model are circular arcs perpendicular to ∂H.

The `"angles"` for Poincaré's model are the usual Euclidean angles between tangents to the circular arcs.

15. | | 10 | For the definition of a hyperbolic distance in the half-plane model, one could again use the cross ratio, as we have done for the Poincaré disk.. But actually, there is another simpler way to get the distance: one requires that the mapping (14) provides an isometry between the half plane and the disk model:

$$ds_D = ds_H \tag{15.1}$$

Recall that the Poinaré disk model has the metric

$$(ds_D)^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} \tag{3}$$

Riemann Metric for Poincaré's Half Plane. *In the Poincaré half plane, the infinitesimal hyperbolic distance* ds *of points with coordinates* (u, v) *and* $(u+du, v+dv)$ *is*

$$(ds_H)^2 = \frac{du^2 + dv^2}{v^2} \tag{15.2}$$

Reason. The metric is determined by the requirement that

$$z = \frac{iw + 1}{w + i} \tag{14}$$

provides an isometry between the half plane and the disk model. Hence we need to convert the known metric (3) of the disk to a metric in the half plane. To calculate the denominator we need to get

$$1 - |z|^2 = \frac{(w+i)(\overline{w} - i) - (iw+1)(-i\overline{w} + 1)}{|w+i|^2} = \frac{2i\overline{w} - 2iw}{|w+i|^2} = \frac{4v}{|w+i|^2}$$

The derivative of the mapping (14) is

$$\frac{dz}{dw} = -\frac{2}{(w+i)^2}$$

Putting the last two formulas into (3) yields

$$\begin{aligned} ds^2 &= \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} = \frac{4|dz|^2}{(1 - |z|^2)^2} \\ &= 4\left|\frac{dz}{dw}\right|^2 |dw|^2 \left(\frac{|w+i|^2}{4v}\right)^2 = 4\left|\frac{2}{(w+i)^2}\right|^2 |dw|^2 \left(\frac{|w+i|^2}{4v}\right)^2 \\ &= \frac{|dw|^2}{v^2} = \frac{du^2 + dv^2}{v^2} \end{aligned}$$

Hence formula (15.2) just ensures the isometry (15.1) between half-plane and the disk model. □

16. | | 10 |

Embedding the Pseudo-sphere into Poincaré's Half Plane. *The pseudo-sphere is sliced along the geodesic in the negative (x, y)-plane, restricting the rotation angle to $-\pi \le \phi < \pi$. The mapping*

$$w = \phi + i\,\frac{a}{r} \tag{16.1}$$

$$u = \phi\,, \quad v = \frac{a}{r} \tag{16.2}$$

maps the (sliced) pseudo-sphere into a half open rectangular domain $\mathcal{P}S_H$ in the upper half plane. The boundary of $\mathcal{P}S_H$ consists of a segment AB with the endpoints $A = -\pi + i, B = \pi + i$, and two unbounded rays $\overrightarrow{A\infty}$ and $\overrightarrow{B\infty}$ with vertices A and B parallel to the positive v axis.

The mapping (16) *transforms the line element $ds_{\mathcal{P}S}$ of the pseudo- sphere to the line element ds_H of the half plane*

$$ds_{\mathcal{P}S} = a\,(ds_H) \tag{16.3}$$

For $a = 1$ and Gaussian curvature $K = -a^{-2} = -1$, we get an isometry. Because an isometry conserves the Gaussian curvature, this shows that the Poincaré half plane has Gaussian curvature -1.

Reason. Use (16.2) (15.2) and (13):

$$u = \phi\,, \quad v = \frac{a}{r} \tag{15}$$

plugged into

$$ds_H{}^2 = \frac{du^2 + dv^2}{v^2} \tag{15.2}$$

One gets

$$ds_H{}^2 = \frac{d\phi^2 + \left(-ar^{-2}\right)^2 dr^2}{a^2 r^{-2}} = a^{-2}\left(r^2 d\phi^2 + \frac{a^2}{r^2} dr^2\right)$$

Now comparing with

$$ds^2_{\mathcal{P}S} = r^2 d\phi^2 + \frac{a^2}{r^2} dr^2 \tag{13}$$

one concludes

$$ds_H{}^2 = a^{-2} (ds_{\mathcal{P}S})^2$$

and hence (16.3) holds. □

17. | | 10 | We now go back to the Poincaré disk model. At first, here are a few remarks about circle-like curves. In hyperbolic geometry, there exist three different types of circle-like curves. I define as a circle-like curve a curve which appears to be a circle in the Poincaré model.

Let ∂D be once more the boundary circle of the Poincaré disk. Take any second circle $\mathcal{C}$. I call its Euclidean center M the quasi-center. The meaning of $\mathcal{C}$ for the hyperbolic geometry of the Poincaré disk depends on the nature of the intersection of the two circles $\mathcal{C}$ and ∂D. There are three important cases:

(i) The circle $\mathcal{C}$ lies totally in the interior D. In that case, it is a circle for hyperbolic geometry. Like in Euclidean geometry, it has a center A. But note that the quasi-center M is different from the center A of $\mathcal{C}$ as an object of hyperbolic geometry.

(ii) The circle $\mathcal{C}$ touches the boundary ∂D from inside, say at endpoint E. In that case, it is a horocycle for hyperbolic geometry. A horocycle has no hyperbolic center, instead it contains an ideal point E. Hence it is unbounded. The hyperbolic circumference of a horocycle is infinite, as follows from part (c) below.

(iii) The circle $\mathcal{C}$ intersects the boundary ∂D at two endpoints E and F. In that case, the circle is either an equidistance line or a geodesic for hyperbolic geometry. A geodesic intersects ∂D perpendicularly. In the case of non perpendicular intersection of ∂D and $\mathcal{C}$, one gets an equidistance line. Actually all points of that equidistance line have the same distance from the hyperbolic straight line with ends E and F.

(a) Take endpoint $Y_+ = (1, 0)$, or i in complex notation, and $O = (0, 0)$. Find the analytic equation for a horocycle $\mathcal{H}$ with apparent diameter OY_+.

Answer. The apparent center is $M = \frac{i}{2}$, and the apparent radius is $\frac{1}{2}$. Hence one gets the equation

$$\left| z - \frac{i}{2} \right|^2 = \frac{1}{4}$$

$$x^2 + \left(y - \frac{1}{2} \right)^2 - \frac{1}{4} = 0$$

$$x^2 + y(y-1) = 0$$

□

(b) We now set up a convenient parametric equation for the horocycle $\mathcal{H}$. Let $Z = (x, y)$ be any point on $\mathcal{H}$ and define the (circumference) angle $\rho \cong \angle OY_+Z$. Calculate $\tan \rho$ in terms of (x, y). Then express x and y in terms of the central angle $2\rho \cong \angle OMZ$. Use double angle formulas, and finally express x and y in terms of $t := \tan \rho$.

Answer.

$$\tan\rho = \frac{x}{1-y} = \frac{y}{x}$$

(17.x) $$x = \frac{\sin 2\rho}{2} = \sin\rho\cos\rho = \frac{\tan\rho}{1+\tan^2\rho} = \frac{t}{1+t^2}$$

(17.y) $$y = \frac{1-\cos 2\rho}{2} = \sin^2\rho = \frac{\tan^2\rho}{1+\tan^2\rho} = \frac{t^2}{1+t^2}$$

□

(c) Calculate the hyperbolic arc length of the arc OZ on the horocycle $\mathcal{H}$. Confirm that the arc length, with zero point at O is just $s = 2t = 2\tan\rho$.

Answer. Differentiation yields

$$x = \frac{t}{1+t^2}\ , \qquad \frac{dx}{dt} = \frac{1-t^2}{(1+t^2)^2}$$

$$y = \frac{t^2}{1+t^2}\ , \qquad \frac{dy}{dt} = \frac{2t}{(1+t^2)^2}$$

$$1 - x^2 - y^2 = 1 - y = \frac{1}{1+t^2} \qquad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{1}{(1+t^2)^2}$$

Hence the hyperbolic metric (3) implies

$$\left(\frac{ds}{dt}\right)^2 = 4(1-x^2-y^2)^{-2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] = 4$$

Hence by elementary integration $s = 2t = 2\tan\rho$. □

18. | | 10 |

Embedding the Pseudo-sphere into Poincaré's Disk Model. *For simplicity, take a pseudo-sphere with* $a = 1$*. This normalizes the Gaussian curvature to be* $K = -1$*. The mapping*

$$z = \frac{r - 1 + ir\phi}{r\phi + i(r+1)} \tag{18}$$

maps the pseudo-sphere isometrically <u>*into*</u> *the Poincaré disk.*

Once more, the pseudo-sphere is sliced along the geodesic in the negative (x, y)*-plane, restricting the rotation angle to* $-\pi \le \phi < \pi$*. This process creates a convenient boundary, which we map to the Poincaré disk via the isometry* (18)*. The image* $\mathcal{PS}_D$ *of the pseudo-sphere is a part of the interior of the horocycle* $\mathcal{H}$ *with apparent diameter* 0 *to* i*.*

Reason. The mapping (18) is constructed as a composition of two mappings $\mathcal{PS} \mapsto H \mapsto D$. Take the mapping $\mathcal{PS} \mapsto H$ given by (16), and the mapping $H \mapsto D$ given by (14). The composition of the mapping

$$w = \phi + i\frac{1}{r} \tag{16}$$

from the pseudo-sphere to the Poincaré half plane, with the mapping

$$z = \frac{iw + 1}{w + i} \tag{14}$$

from the Poincaré half plane to the Poincaré disk is the following mapping

$$z = \frac{i\left(\phi + \frac{i}{r}\right) + 1}{\phi + \frac{i}{r} + i} = \frac{r - 1 + ir\phi}{r\phi + i(r+1)} \tag{18}$$

from the pseudo-sphere to the Poincaré disk. Both factors (16) and (14) are isometries, as stated by formulas

$$ds_{\mathcal{PS}} = ds_H \tag{16.3}$$

and

$$ds_D = ds_H \tag{15.1}$$

Hence their composition (18) conserves the line element: $ds_{\mathcal{PS}} = ds_H = ds_D$. In other words, the mapping (18) is an isometry, too. □

19. | | 10 | On the pseudo-sphere, we use as parameters the cylindrical coordinates r and ϕ, as explained in part 13. The boundary of the pseudo-sphere is given by $r = 1$, and $-\pi < \phi < \pi$.

(a) To which curve in the disk D is the boundary mapped by the isometry (18)?

Answer. The boundary $\partial \mathcal{P}S_D$ of image $\mathcal{P}S_D$ consists of three circular arcs. A segment AB of a horocycle $\mathcal{H}$ with endpoints

$$A = \frac{i\pi}{\pi + 2i}, \quad B = \frac{i\pi}{\pi - 2i}$$

as well as two geodesic rays $\overrightarrow{AY_+}$ and $\overrightarrow{BY_+}$ with vertices A and B pointing to the ideal endpoint $Y_+ = i$. □

(b) Give a parametric equation for the boundary, with parameter ϕ, at first in complex notation for $z = x + iy$. Then separate into real and imaginary parts to get equations for x and y.

Answer. Simply put $r = 1$ into (18). One gets

$$z = \frac{i\phi}{\phi + 2i}$$

To separate real and imaginary parts, one needs a make the denominator real:

$$z = \frac{i\phi}{\phi + 2i} = \frac{i\phi(\phi - 2i)}{(\phi + 2i)(\phi - 2i)}$$

$$x + iy = \frac{i\phi^2 + 2\phi}{\phi^2 + 4} \tag{19}$$

$$x = \frac{2\phi}{\phi^2 + 4} \tag{19.x}$$

$$y = \frac{\phi^2}{\phi^2 + 4} \tag{19.y}$$

□

(c) Check that your parametric equation is a circle with center $\frac{i}{2}$.

Answer. This is a parametric equation of a circle with center $\frac{i}{2}$ because

$$z - \frac{i}{2} = \frac{i\phi}{\phi + 2i} - \frac{i}{2} = \frac{i(\phi - 2i)}{2(\phi + 2i)}$$

$$\left| z - \frac{i}{2} \right| = \frac{1}{2}$$

□

(d) Compare (19) with the result

$$x = \frac{\tan\rho}{1+\tan^2\rho} = \frac{t}{1+t^2} \tag{17.x}$$

$$y = \frac{\tan^2\rho}{1+\tan^2\rho} = \frac{t^2}{1+t^2} \tag{17.y}$$

and check that $\phi = 2t$. Hence, because of $s = 2t$ was shown in item 17(c), one concludes that $\phi = s = 2t = 2\tan\rho$. Of course $\phi = s$ follows directly because of the isometries $\mathcal{P}S \mapsto H \mapsto D$.

(e) Use part (d) and explain in a drawing, how you use $s = 2\tan\rho$ to measure the arclength on the horocycle $\mathcal{H}$.

Draw sketches of the pseudo-sphere as it appears in the domains $\mathcal{P}S \mapsto H \mapsto D$. Use different colors for the different vertices and edges of the boundary, but same colors for corresponding objects in all three domains $\mathcal{P}S \mapsto H \mapsto D$, as they are mapped by our isometries from above.

20. | | 10 | The Poincaré disk can be tiled with congruent triangles. Indeed, there exist infinitely many different types of such tilings. I choose a tiling with congruent equilateral triangles, such that at each vertex seven triangles meet.

(a) Use Gauss' remarkable theorem to calculate the hyperbolic area of one such triangle.

Solution. I measure angles in radian measure. The angles of one triangle are all

$$\alpha = \beta = \gamma = \frac{2\pi}{7}$$

and hence the defect of the angle sum is

$$\alpha + \beta + \gamma - \pi = \frac{6\pi}{7} - \pi = -\frac{\pi}{7}$$

Since the Gaussian curvature of Poincaré's model is $K = -1$, the area is just the negative of the excess, (this is also called the defect) and is $\frac{\pi}{7}$. □

(b) Overlay two drawings, of the tiling by equilateral triangles, and a second drawing of the image $\mathcal{P}S_D$ of the pseudo- sphere and its boundary $\partial\mathcal{P}S_D$.

How many of those triangles of the tiling fit entirely into $\mathcal{P}S_D$?

How many triangles make up (with bids and pieces!) the total area of $\mathcal{P}S_D$?

Solution. □

Solution. Using item 12, one calculated that the total area of the pseudo-sphere is 2π. This equals the area of 14 triangles. □