The Fermat point

Let $ABC$ be a triangle whose internal angles are all less than $120^\circ$. We define its Fermat point as the point $P$ satisfying $\angle APB = \angle BPC = \angle CPA = 120^\circ$. Using this definition, the Fermat point clearly exist since it is the intersection of the (open) arc $\{Q : \angle AQB = 120^\circ\}$ with the (open) arc $\{Q : \angle BQC = 120^\circ\}$.

Construct the regular triangles $A'B'C_\Delta$, $AB'C_\Delta$, and $ABC'_\Delta$ as shown in Fig. 1.

![Figure 1: Constructing the Fermat point using regular triangles](image)

**Proposition 1** The lines $AA'$, $BB'$ and $CC'$ meet in the Fermat point of the triangle.

**Proof:** Rotation by $60^\circ$ around $A$ sends $AB'B_\Delta$ into $ACC'_\Delta$. Introducing $P$ for the intersection of $B'B$ and $C'C$ we see that $\angle CPB' = 60^\circ$ and so $\angle CPB = 120^\circ$. Let us introduce $P_1$ for the image of $P$ under this rotation. Then $|P_1C'| = |PB|$ by the congruence induced by the rotation, and $|PP_1| = |PA|$ since $PP_1A_\Delta$ is a regular triangle. Therefore

$$|CC'| = |CP| + |PP_1| + |P_1C| = |PA| + |PB| + |PC|.$$

Consider now the rotation by $60^\circ$ around $B$. This takes $BC'C_\Delta$ into $BAA'_\Delta$ and let us denote the image of $P$ under this rotation by $P_2$. (Note that we know only that $P$ is on $CC'$, we don’t know yet whether it is also on $AA'$.) By the congruence $BC'C_\Delta \cong BAA'_\Delta$ we have $|AA'| = |CC'|$ and so $|AA'| = |PA| + |PB| + |PC|$. Observe also that $|P_2A| = |PC|$ by the congruence induced by the rotation and that $|PP_2| = |PB|$ since $PP_2B_\Delta$ is a regular triangle. Therefore we get that

$$|A'P_2| + |P_2P| + |PA| = |PC| + |PB| + |PA| = |A'A|.$$

If $P$ or $P_2$ is not on the line $AA'$ then (by the triangle inequality), the sum $|A'P_2| + |P_2P| + |PA|$ is strictly greater than $|A'A|$, in contradiction with the above equality. Thus $P$ is also on the line $AA'$. Now $\angle CPA = 120^\circ$ follows in analogy with $\angle CPB = 120^\circ$. $\diamond$
The first half of the proof of Proposition 1 is almost repeated in the proof of the following characterization of the Fermat point.

**Proposition 2** *The Fermat point minimizes the sum $|QA| + |QB| + |QC|$ among all points $Q$ in the plane.*

**Proof:** Consider any point $Q$ in the plane that is different from the Fermat point. W.l.o.g. we may assume that $\angle AQC \neq 120^0$. Let us rotate $QAC\triangle$ around $A$ by $60^0$, as shown in Fig. 2. Denote the image of $Q$ and $B$ respectively by $Q'$ and $C'$ respectively. By rotational congruence we have $|QB| = |Q'C'|$ and we also have $|QQ'| = |AQ'|$ since $AQQ'\triangle$ is regular. Thus

$$|QA| + |QB| + |QC| = |CQ| + |QQ'| + |Q'C|.$$

Since $\angle AQC \neq 120^0$, the angle $\angle AQQ' \neq 180^0$ and, by the triangle inequality, $|CQ| + |QQ'| + |Q'C|$ is strictly more than $|CC'|$. In the proof of the previous proposition we have seen that the Fermat point satisfies $|PA| + |PB| + |PC| = |CC'|$. 

\[\diamond\]