1. (B) The average changes from $\dfrac{87+83+88}{3} = 86$ to $\dfrac{87+83+88+90}{4} = 87$, an increase of 1.

2. (D) Square both sides of the given equation to obtain $2+\sqrt{x} = 9$. Thus $\sqrt{x} = 7$, and $x = 49$, which satisfies the given equation.

3. (B) The total price advertised on television is

$$\$29.98 + \$29.98 + \$29.98 + \$9.98 = \$99.92,$$

so this is $\$99.99 - \$99.92 = \$0.07$ less than the in-store price.

OR

The three payments are each 2 cents less than \$30, and the shipping & handling charge is 2 cents less than \$10, so the total price advertised on television is 8 cents less than \$100. The total in-store price is 1 cent less than \$100, so the amount saved by buying the appliance from the television advertiser is 7 cents.

4. (B) Since $M = 0.3Q = 0.3(0.2P) = 0.06P$ and $N = 0.5P$, we have

$$\frac{M}{N} = \frac{0.06P}{0.5P} = \frac{6}{50} = \frac{3}{25}\ .$$

5. (C) The number of ants is approximately the product

$$(300 \text{ ft}) \times (400 \text{ ft}) \times (12 \text{ in/ft})^2 \times (3 \text{ ants/in}^2) = 300 \times 400 \times 144 \times 3 \text{ ants},$$

which is $3 \times 4 \times 1.44 \times 3 \times 10^{2+2+2} \approx 50 \times 10^6$.

6. (C) Think of A as the bottom. Fold B up to be the back. Then x folds upward to become the left side and C folds forward to become the right side, so C is opposite x.

7. (C) The length of the flight path is approximately the circumference of Earth at the equator, which is

$$C = 2\pi \cdot 4000 = 8000\pi \text{ miles}.$$

The time required is

$$\frac{8000\pi}{500} = 16\pi \begin{cases} > 16(3.1) = 49.6 \text{ hours} \\ < 16(3.2) = 51.2 \text{ hours}, \end{cases}$$

so the best choice is 50 hours. **Query.** What is a negligible height; i.e., for which heights above the equator would the flight-time be closer to choice **(C)** than to **(D)**?

8. (C) Because $\triangle ABC$ is a right triangle, the *Pythagorean Theorem* implies that $BA = 10$. Since $\triangle DBE \sim \triangle ABC$,

$$\frac{BD}{BA} = \frac{DE}{AC}. \qquad \text{So} \qquad BD = \frac{DE}{AC}(BA) = \frac{4}{6}(10) = \frac{20}{3}.$$

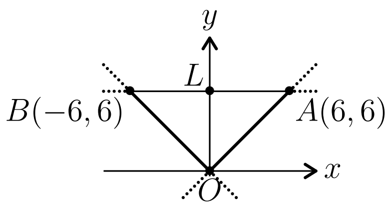
OR

Since $\sin B = \dfrac{DE}{BD}$, we have $BD = \dfrac{DE}{\sin B}$. Moreover, $BA = 10$ by the *Pythagorean Theorem*, so $\sin B = \dfrac{AC}{BA} = \dfrac{3}{5}$. Hence $BD = \dfrac{4}{3/5} = \dfrac{20}{3}$.

9. (D) Since all the acute angles in the figure measure 45°, all the triangles must be isosceles right triangles. It follows that all the triangles must enclose one, two or four of the eight small triangular regions. Besides the eight small triangles, there are four triangles that enclose two of the small triangular regions and four triangles that enclose four, making a total of 16.

10. (E) Let O be the origin, and let A and B denote the points where $y = 6$ intersects $y = x$ and $y = -x$ respectively. Let $\overline{OL}$ denote the altitude to side $\overline{AB}$ of $\triangle OAB$. Then $OL = 6$. Also, $AL = BL = 6$. Thus, the area of $\triangle OAB$ is

$$\frac{1}{2}(AB)(OL) = \frac{1}{2} \cdot 12 \cdot 6 = 36.$$



OR

Let $A' = (6, 0)$. Then $\triangle A'OA \cong \triangle LOB$, so the area of triangle AOB equals the area of square $A'OLA$, which is $6^2 = 36$.

OR

Use the determinant formula for the area of the triangle: $\dfrac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 6 & 6 & 1 \\ -6 & 6 & 1 \end{vmatrix} = 36.$

11. (C) Condition (i) requires that a be one of the two digits, 4 or 5. Condition (ii) requires that d be one of the two digits, 0 or 5. Condition (iii) requires that the ordered pair (b, c) be one of these six ordered pairs:

$$(3,4),\ (3,5),\ (3,6),\ (4,5),\ (4,6),\ (5,6).$$

Therefore, there are $2 \times 2 \times 6 = 24$ numbers N satisfying the conditions.

12. (D) Since f is a linear function, it has the form $f(x) = mx + b$. Because $f(1) \le f(2)$, we have $m \ge 0$. Similarly, $f(3) \ge f(4)$ implies $m \le 0$. Hence, $m = 0$, and f is a constant function. Thus, $f(0) = f(5) = 5$.

13. (C) The addition in the columns containing the ten-thousands and hundred-thousands digits is incorrect. The only digit common to both these columns is 2. Changing these 2's to 6's makes the arithmetic correct. Changing the other two 2's to 6's has no effect on the correctness of the remainder of the addition, and no digit other than 2 could be changed to make the addition correct. Thus, $d = 2$, $e = 6$, and $d + e = 8$.

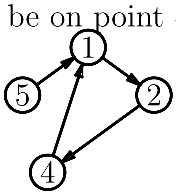
14. (E) Since $f(3) = a(3)^4 - b(3)^2 + 3 + 5$ and $f(-3) = a(-3)^4 - b(-3)^2 - 3 + 5$, it follows that $f(3) - f(-3) = 6$. Thus, $f(3) = f(-3) + 6 = 2 + 6 = 8$.

Note. For any x, $f(x) - f(-x) = 2x$, so $f(x) = f(-x) + 2x$.

OR

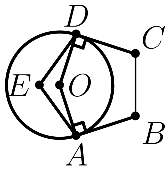
Since $2 = f(-3) = 81a - 9b - 3 + 5$ we have $b = 9a$. Thus $f(3) = 81a - 9b + 3 + 5 = 81a - 9(9a) + 8 = 8$.

15. (D) With the first jump, the bug moves to point 1, with the second to 2, with the third to 4 and with the fourth it returns to 1. Thereafter, every third jump it returns to 1. Thus, after $n > 0$ jumps, the bug will be on 1, 2 or 4, depending on whether n is of the form $3k + 1$, $3k + 2$ or $3k$, respectively. Since $1995 = 3(665)$, the bug will be on point 4 after 1995 jumps.



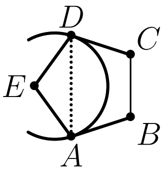
16. (E) Let A denote the number in attendance in Atlanta, and let B denote the number in attendance in Boston. We are given $45,000 \le A \le 55,000$ and $0.9B \le 60,000 \le 1.1B$, so $54,546 \le B \le 66,666$. Hence the largest possible difference between A and B is $66,666 - 45,000 = 21,666$, so the correct choice is **(E)**.

17. (E) Let O be the center of the circle. Since the sum of the interior angles in any n-gon is $(n-2)180°$, the sum of the angles in $ABCDO$ is $540°$. Since $\angle ABC = \angle BCD = 108°$ and $\angle OAB = \angle ODC = 90°$, it follows that the measure of $\angle AOD$, and thus the measure of minor arc AD, equals $144°$.



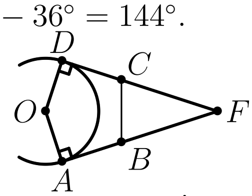
OR

Draw $\overline{AD}$. Since $\triangle AED$ is isosceles with $\angle AED = 108°$, it follows that $\angle EDA = \angle EAD = 36°$. Consequently, $\angle ADC = 108° - 36° = 72°$. Since $\angle ADC$ is a tangent-chord angle for the arc in question, the measure of the arc is $2(72°) = 144°$.



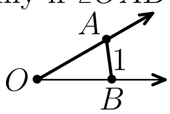
OR

Let O be the center of the circle, and extend $\overline{DC}$ and $\overline{AB}$ to meet at F. Since $\angle DCB = 108°$ and $\triangle BCF$ is isosceles, it follows that $\angle AFD = [180° - 2(180° - 108°)] = 36°$. Since $\angle ODF = \angle OAF = 90°$, in quadrilateral $OAFD$ we have angles AOD and AFD supplementary, so the measures of angle AOD and the minor arc AD are $180° - 36° = 144°$.



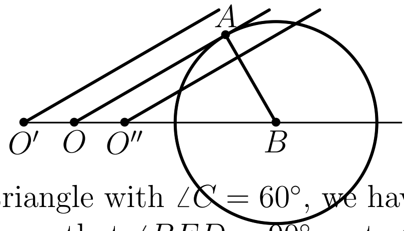
Note. A circle can be drawn tangent to two intersecting lines at given points on those lines if and only if those points are equidistant from the point of intersection of the lines.

18. (D) By the *Law of Sines*, $\dfrac{OB}{\sin\angle OAB} = \dfrac{AB}{\sin\angle AOB} = \dfrac{1}{1/2}$, so $OB = 2\sin\angle OAB \le 2\sin 90° = 2$, with equality if and only if $\angle OAB = 90°$.



OR

Consider B to be fixed on a ray originating at a variable point O, and draw another ray so the angle at O is $30°$. A possible position for A is any intersection of this ray with the circle of radius 1 centered at B. The largest value for OB for which there *is* an intersection point A occurs when $\overline{OA}$ is tangent to the circle. Since $\triangle OBA$ is a 30°-60°-90° triangle with $AB = 1$, it follows that $OB = 2$ is largest.

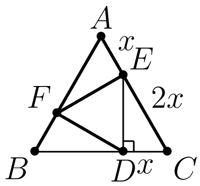


19. (C) Since CDE is a right triangle with $\angle C = 60°$, we have $CE = 2DC$. Also, $\angle BFD = 90° = \angle FEA$. To see that $\angle BFD = 90°$, note that

$$\angle BDF + \angle FDE + 90° = \angle BDF + 60° + 90° = 180°.$$

Thus $\angle BDF = 30°$ and since $\angle DBF = 60°$, $\angle BFD = 90°$. That $\angle FEA = 90°$ follows similarly. Since $\triangle DEF$ is equilateral, the three small triangles are congruent and $AE = DC$. Let $AC = 3x$. Then $EC = 2x$ and $DE = \sqrt{3}x$. The desired ratio is

$$\left(\frac{DE}{AC}\right)^2 = \left(\frac{\sqrt{3}x}{3x}\right)^2 = \frac{1}{3}.$$



20. (B) The quantity $ab+c$ will be even if ab and c are both even or both odd. Furthermore, ab will be odd only when both a and b are odd, so the probability of ab being odd is $\frac{3}{5} \cdot \frac{3}{5} = \frac{9}{25}$. Thus the probability of ab being even is $1 - \frac{9}{25} = \frac{16}{25}$. Hence, the required probability is $\frac{16}{25} \cdot \frac{2}{5} + \frac{9}{25} \cdot \frac{3}{5} = \frac{59}{125}$.

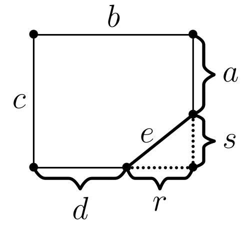
21. (E) The diagonals of a rectangle are of the same length and bisect each other. The given diagonal has length $\sqrt{(-4-4)^2 + (-3-3)^2} = 10$ and midpoint $(0,0)$. The other diagonal must have end points on the circle of radius 5 centered at the origin and must have integer coordinates for each end point. We must find integer solutions to $x^2 + y^2 = 5^2$. The only possible diagonals, other than the given diagonal, are the segments: $\overline{(0,5)\,(0,-5)}$, $\overline{(5,0)\,(-5,0)}$, $\overline{(3,4)\,(-3,-4)}$, $\overline{(-3,4)\,(3,-4)}$, $\overline{(4,-3)\,(-4,3)}$. Each of these five, with the original diagonal, determines a rectangle.

22. (E) Let the sides of the pentagon be a, b, c, d and e, and let r and s be the legs of the triangular region cut off as shown. The equation $r^2 + s^2 = e^2$ has no solution in positive integers when $e = 19$ or $e = 31$. Therefore, e equals 13,

20 or 25, and the possibilities for $\{r, s, e\}$ are the well-known Pythagorean triples

$$\{5,12,13\}, \quad \{12,16,20\}, \quad \{15,20,25\}, \quad \{7,24,25\}.$$

Since 16, 15 and 24 do not appear among any of the pairwise differences of $\{13,19,20,25,31\}$, the only possibility is $\{5,12,13\}$. Then $a = 19$, $b = 25$, $c = 31$, $d = 20$ and $e = 13$. Hence, the area of the pentagon is $31 \times 25 - \frac{1}{2}(12 \times 5) = 775 - 30 = 745$.



23. (D) Since the longest side of a triangle must be less than the sum of the other two sides, it follows that $4 < k < 26$. For the triangle to be obtuse, either $11^2 + 15^2 < k^2$, or $11^2 + k^2 < 15^2$. Therefore the 13 suitable values of k are 5, 6, 7, 8, 9, 10, 19, 20, 21, 22, 23, 24 and 25.

24. (A) Note that

$$C = A\log_{200} 5 + B\log_{200} 2 = \log_{200} 5^A + \log_{200} 2^B = \log_{200}\left(5^A \cdot 2^B\right),$$

so $200^C = 5^A \cdot 2^B$. Therefore, $5^A \cdot 2^B = 200^C = (5^2 \cdot 2^3)^C = 5^{2C}2^{3C}$. By uniqueness of prime factorization, [1] $A = 2C$ and $B = 3C$. Letting $C = 1$ we get $A = 2$, $B = 3$ and $A + B + C = 6$. The triplet $(A, B, C) = (2, 3, 1)$ is the only solution with no common factor greater than 1.

25. (B) Since the median and mode are both 8 and the range is 18, the list must take on one of these two forms:

	(I) :	$a, b, 8, 8, a{+}18$	where $a \le b \le 8 \le a + 18$
or	(II) :	$c, 8, 8, d, c{+}18$	where $c \le 8 \le d \le c + 18$.

The sum of the five integers must be 60, since their mean is 12. In case (I), the requirement that $2a + b + 34 = 60$ contradicts $a, b \le 8$. In case (II), $2c + d + 34 = 60$ and $c \le 8 \le d \le c + 18$ lead to these six pairs, (c, d):

$$(8,10),\ (7,12),\ (6,14),\ (5,16),\ (4,18),\ (3,20).$$

Thus, the second largest entry in the list can be any of the six numbers $d = 10, 12, 14, 16, 18, 20$.

[1] An application of the *Fundamental Theorem of Arithmetic.*

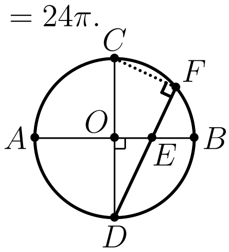
26. (C) Draw segment $\overline{FC}$. Angle CFD is a right angle since arc CFD is a semicircle. Then right triangles DOE and DFC are similar, so

$$\frac{DO}{DF} = \frac{DE}{DC}.$$

Let $DO = r$ and $DC = 2r$. Substituting, we have

$$fracr8 = \frac{6}{2r}, \quad 2r^2 = 48, \quad r^2 = 24.$$

Then the area of the circle is $\pi r^2 = 24\pi$.

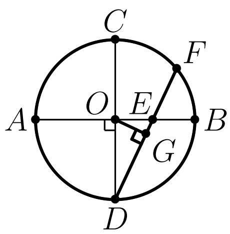


OR

Let $OA = OB = r$ and $OE = x$. Substituting into $AE \cdot EB = DE \cdot EF$ gives $(r + x)(r - x) = 6 \cdot 2$ so $r^2 - x^2 = 12$. In right triangle EOD, $r^2 + x^2 = 36$. Add to find $2r^2 = 48$. Thus, the area of the circle is $\pi r^2 = 24\pi$.

OR

Construct $\overline{OG} \perp \overline{DF}$ with G on $\overline{DF}$. Then $DG = \frac{1}{2}DF = 4$. Since $\overline{OG}$ is an altitude to the hypotenuse of right triangle EOD, we have $\dfrac{DE}{DO} = \dfrac{DO}{DG}$. Let $DO = r$. Then $\dfrac{6}{r} = \dfrac{r}{4}$, so $r^2 = 24$, and the area of the circle is $\pi r^2 = 24\pi$.



27. (E) Calculating the first five values of f,

$$f(1) = 0, \quad f(2) = 2, \quad f(3) = 6, \quad f(4) = 14, \quad f(5) = 30,$$

we are led to the conjecture that $f(n) = 2^n - 2$. We prove this by induction: Observe that each of the interior numbers in row n is used twice and each of the end numbers is used once as a term in computing the interior terms of row $n+1$; i.e.,

$$f(n+1) = [2f(n) - 2(n-1)] + 2n = 2f(n) + 2,$$

so if $f(n) = 2^n - 2$, then $f(n+1) = 2f(n) + 2 = 2(2^n - 2) + 2 = 2^{n+1} - 2$. Therefore, we seek the remainder when $f(100) = 2^{100} - 2$ is divided by 100. Use the fact that 76^2 has remainder 76 when divided by 100.[2]**Query:** What other positive integers N have the property that N^2 has remainder N when divided by 100? We find

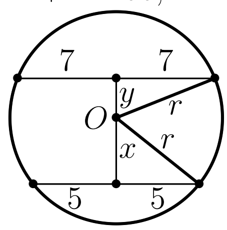
$$\begin{aligned} 2^{10} &= 100K + 24, \\ 2^{20} &= 100L + 76, \\ 2^{40} &= 100M + 76, \\ 2^{80} &= 100N + 76, \\ 2^{100} &= 100Q + 76, \end{aligned}$$

for positive integers K, L, M, N, Q, so $f(100) = 2^{100} - 2$ has remainder 74 when divided by 100.

28. (E) Let x be the distance from the center O of the circle to the chord of length 10, and let y be the distance from O to the chord of length 14. Let r be the radius. Then,

$$\begin{aligned} x^2 + 25 &= r^2, \\ y^2 + 49 &= r^2, \\ \text{so} \qquad x^2 + 25 &= y^2 + 49. \\ \text{Therefore,} \qquad x^2 - y^2 &= (x-y)(x+y) = 24. \end{aligned}$$

If the chords are on the same side of the center of the circle, $x - y = 6$. If they are on opposite sides, $x + y = 6$. But $x - y = 6$ implies that $x + y = 4$, which is impossible. Hence $x + y = 6$ and $x - y = 4$. Solve these equations simultaneously to get $x = 5$ and $y = 1$. Thus, $r^2 = 50$, and the chord parallel to the given chords and midway between them is 2 units from the center. If the chord is of length $2d$, then $d^2 + 4 = 50$, $d^2 = 46$, and $a = (2d)^2 = 184$.



OR

2†

The diameter perpendicular to the chords is divided by the chord of length $\sqrt{a}$ into segments with lengths c and d as shown. Then

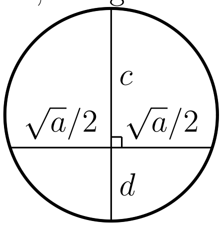
$$cd = \left(\frac{\sqrt{a}}{2}\right)^2 = \frac{a}{4}.$$

Treat the chords 3 units above and 3 units below similarly:

$$\begin{aligned}(c-3)(d+3) &= \left(\frac{14}{2}\right)^2 \\ (c+3)(d-3) &= \left(\frac{10}{2}\right)^2.\end{aligned}$$

Adding the last two equations, we get $2cd - 18 = 49 + 25 = 74$. Thus,

$2cd = 92$ so $a = 4cd = 184$.



29. (C) Since the three factors, a, b and c, must be distinct, we seek the number of positive integer solutions to

$$abc = 2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11, \qquad \text{with} \qquad a < b < c.$$

The prime factors of a, b and c must be disjoint subsets of $S = \{2, 3, 5, 7, 11\}$, no more than one subset can be empty, and the union of the subsets must be S. The numbers of elements in the subsets can be: $0, 1, 4$; $0, 2, 3$; $1, 1, 3$; or $1, 2, 2$.

In the $0, 1, 4$ case, there are 5 ways to choose three subsets with these sizes.
In the $0, 2, 3$ case, there are $\binom{5}{2} = 10$ ways to choose the three subsets.
In the $1, 1, 3$ case, there are $\binom{5}{3} = 10$ ways to choose the three subsets.
In the $1, 2, 2$ case, there are 5 ways of choosing the one-element subset and $\frac{1}{2} \cdot \binom{4}{2} = 3$ ways of dividing the remaining four elements into two subsets of two elements each, yielding 15 ways of choosing the three subsets in this case.

Thus there are a total of $5 + 10 + 10 + 15 = 40$ ways of choosing our three subsets and, therefore, 40 ways of expressing 2310 in the required manner. Since factorization into primes is unique, these 40 triplets of sets give distinct solutions.

OR

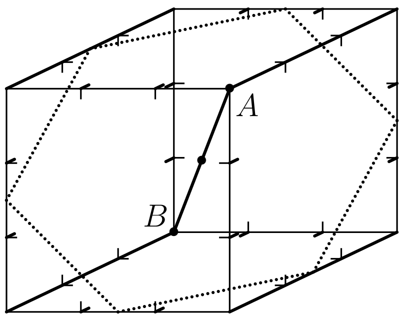
There are $3^5 = 243$ ordered triples, (a, b, c), of integers such that $abc = 2310$, since each of the five prime factors of $2310 = 2{\cdot}3{\cdot}5{\cdot}7{\cdot}11$ divides exactly one of a, b or c. In three of these 243 ordered triples, two of a, b, c equal 1. In the remaining 240 ordered triples, a, b and c are distinct, since 2310 is square-free. Each unordered triple whose product is 2310 is represented by $3! = 6$ of the 240 ordered triples (a, b, c), so the answer is $240/6 = 40$.

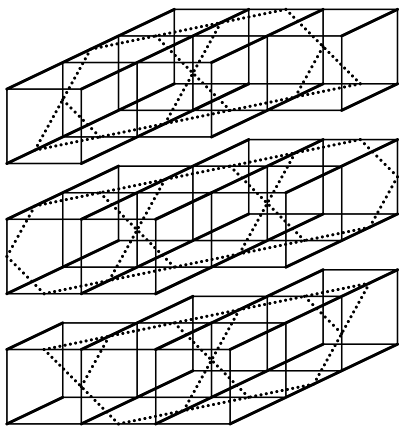
30. (D) Suppose the coordinates of the vertices of the unit cubes occur at (i, j, k) for all $i, j, k \in \{0, 1, 2, 3\}$. The equation of the plane that bisects the large cube's diagonal from $(0, 0, 0)$ to $(3, 3, 3)$ is $x + y + z = 9/2$. That plane meets a unit cube if and only if the ends of the unit cube's diagonal from (i, j, k) to $(i{+}1, j{+}1, k{+}1)$ lie on opposite sides of the plane. Therefore, this problem is equivalent to counting the number of the 27 triples (i, j, k) with $i, j, k \in \{0, 1, 2\}$ for which $i + j + k < 4.5 < i + j + k + 3$. Only 8 of these 27 triples do not satisfy these inequalities:

$$(0, 0, 0),\ (1, 0, 0),\ (0, 1, 0),\ (0, 0, 1),\ (1, 2, 2),\ (2, 1, 2),\ (2, 2, 1),\ (2, 2, 2).$$

Therefore, $27 - 8 = 19$ of the unit cubes are intersected by the plane.

A sketch can help you visualize the 19 unit cubes intersected by the plane. Suppose the plane is perpendicular to the interior diagonal $\overline{AB}$ at its midpoint. That plane intersects the surface of the large cube in a regular hexagon.





The sketch shows that nineteen of the twenty-seven unit cubes are intersected by this plane, with six each in the bottom and top layers and seven in the middle layer. The corner unit cube at vertex A and the three unit cubes adjacent to it are missed by this plane, as are the four symmetric to these at vertex B.