1. **(D)** The mistake occurs in the tens column where any of the digits of the addends can be decreased by 1 or the 5 in the sum changed to 6 to make the addition correct. The largest of these digits is 7.

2. **(A)** Walter gets an extra \$2 per day for doing chores exceptionally well. If he never did them exceptionally well, he would get \$30 for 10 days of chores. The extra \$6 must be for 3 days of exceptional work.

3. **(E)** $\frac{(3!)!}{3!} = \frac{6!}{3!} = 6 \cdot 5 \cdot 4 = 120.$

OR

$\frac{(3!)!}{3!} = \frac{6!}{6} = 5! = 5 \cdot 4 \cdot 3 \cdot 2 = 120.$

4. **(D)** The largest possible median will occur when the three numbers not given are larger than those given. Let a, b, and c denote the three missing numbers, where $9 \le a \le b \le c$. Ranked from smallest to largest, the list is

$$3, 5, 5, 7, 8, 9, a, b, c,$$

so the median value is 8.

5. **(E)** The largest fraction is the one with largest numerator and smallest denominator. Choice **(E)** has both.

6. **(E)** Since $0^z = 0$ for any $z > 0$, $f(0) = f(-2) = 0$. Since $(-1)^0 = 1$,

$$f(0) + f(-1) + f(-2) + f(-3) = (-1)^0(1)^2 + (-3)^{-2}(-1)^0 = 1 + \frac{1}{(-3)^2} = \frac{10}{9}.$$

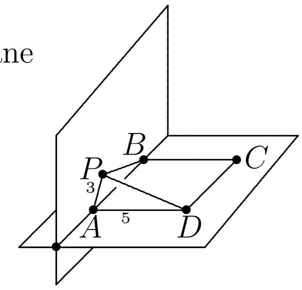
7. **(B)** The sum of the children's ages is 10 because \$9.45 − \$4.95 = \$4.50 = 10 × \$0.45. If the twins were 3 years old or younger, then the third child would not be the youngest. If the twins are 4, the youngest child must be 2.

8. **(D)** Since $3 = k \cdot 2^r$ and $15 = k \cdot 4^r$, we have

$$5 = \frac{15}{3} = \frac{k \cdot 4^r}{k \cdot 2^r} = \frac{2^{2r}}{2^r} = 2^r.$$

Thus, by definition, $r = \log_2 5$.

9. **(B)** Since line segment AD is perpendicular to the plane of PAB, angle PAD is a right angle. In right triangle PAD, $PA = 3$ and $AD = AB = 5$. By the Pythagorean Theorem, $PD = \sqrt{3^2+5^2} = \sqrt{34}$. The fact that $PB = 4$ was not needed.

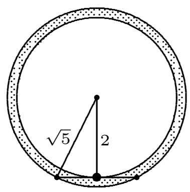


10. **(D)** There are 12 edges, 12 face diagonals, and 4 space diagonals for a total of $12+12+4=28$.

OR

Each pair of vertices of the cube determines a line segment. There are $\binom{8}{2} = \frac{8!}{(8-2)!2!} = \frac{8\cdot 7}{2} = 28$ such pairs.

11. **(D)** The endpoints of each of these line segments are at distance $\sqrt{2^2+1^2} = \sqrt{5}$ from the center of the circle. The region is therefore an annulus with inner radius 2 and outer radius $\sqrt{5}$. The area covered is $\pi(\sqrt{5})^2 - \pi(2)^2 = \pi$.



Note. The area of the annular region covered by the segments of length 2 does not depend on the radius of the circle.

12. **(B)** Since k is odd, $f(k) = k+3$. Since $k+3$ is even,

$$f(f(k)) = f(k+3) = (k+3)/2.$$

If $(k+3)/2$ is odd, then

$$27 = f(f(f(k))) = f((k+3)/2) = (k+3)/2 + 3,$$

which implies that $k = 45$. This is not possible because $f(f(f(45))) = f(f(48)) = f(24) = 12$. Hence $(k+3)/2$ must be even, and

$$27 = f(f(f(k))) = f((k+3)/2) = (k+3)/4,$$

which implies that $k = 105$. Checking, we find that

$$f(f(f(105))) = f(f(108)) = f(54) = 27.$$

Hence the sum of the digits of k is $1+0+5=6$.

13. **(D)** Let x be the number of meters that Moonbeam runs to overtake Sunny, and let r and mr be the rates of Sunny and Moonbeam, respectively. Because Sunny runs $x-h$ meters in the same time that Moonbeam runs x meters, it follows that $\dfrac{x-h}{r}=\dfrac{x}{mr}$. Solving for x, we get $x=\dfrac{hm}{m-1}$.

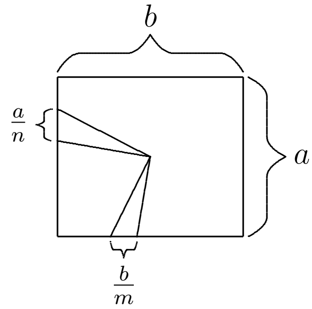
14. **(C)** Since $E(100)=E(00)$, the result is the same as $E(00)+E(01)+E(02)+E(03)+\cdots+E(99)$, which is the same as

$$E(00010203\ldots 99).$$

There are 200 digits, and each digit occurs 20 times, so the sum of the even digits is $20(0+2+4+6+8)=20(20)=400$.

15. **(B)** Let the base of the rectangle be b and the height a. Triangle A has an altitude of length $b/2$ to a base of length a/n, and triangle B has an altitude of length $a/2$ to a base of length b/m. Thus the required ratio of areas is

$$\frac{\frac{1}{2}\cdot\frac{a}{n}\cdot\frac{b}{2}}{\frac{1}{2}\cdot\frac{b}{m}\cdot\frac{a}{2}}=\frac{m}{n}.$$



16. **(D)** There are 15 ways in which the third outcome is the sum of the first two outcomes.

$\boxed{(1,1,2)}$	$\boxed{(2,1,3)}$	$(3,1,4)$	$(4,1,5)$	$(5,1,6)$
$\boxed{(1,2,3)}$	$\boxed{(2,2,4)}$	$\boxed{(3,2,5)}$	$\boxed{(4,2,6)}$	
$(1,3,4)$	$\boxed{(2,3,5)}$	$(3,3,6)$		
$(1,4,5)$	$\boxed{(2,4,6)}$			
$(1,5,6)$				

Since the three tosses are independent, all of the 15 possible outcomes are equally likely. At least one "2" appears in exactly eight of these outcomes, so the required probability is $8/15$.

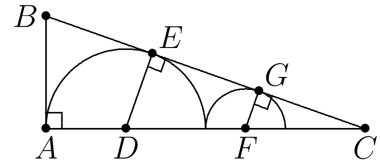
17. **(E)** In the 30°-60°-90° triangle CEB, $BC=6\sqrt{3}$. Therefore, $FD=AD-AF=6\sqrt{3}-2$. In the 30°-60°-90° triangle CFD, $CD=FD\sqrt{3}=18-2\sqrt{3}$. The area of rectangle $ABCD$ is

$$(BC)(CD)=\left(6\sqrt{3}\right)\left(18-2\sqrt{3}\right)=108\sqrt{3}-36\approx 151.$$

18. **(D)** Let D and F denote the centers of the circles. Let C and B be the points where the x-axis and y-axis intersect the tangent line, respectively. Let E and G denote the points of tangency as shown. We know that $AD = DE = 2$, $DF = 3$, and $FG = 1$. Let $FC = u$ and $AB = y$. Triangles FGC and DEC are similar, so

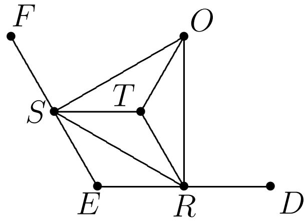
$$\frac{u}{1} = \frac{u+3}{2},$$

which yields $u = 3$. Hence, $GC = \sqrt{8}$. Also, triangles BAC and FGC are similar, which yields



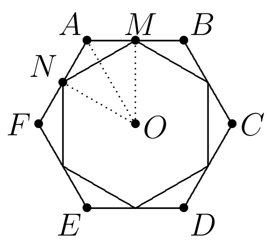
$$\frac{y}{1} = \frac{BA}{FG} = \frac{AC}{GC} = \frac{8}{\sqrt{8}} = \sqrt{8} = 2\sqrt{2}.$$

19. **(D)** Let R and S be the vertices of the smaller hexagon adjacent to vertex E of the larger hexagon, and let O be the center of the hexagons. Then, since $\angle ROS = 60°$, quadrilateral $ORES$ encloses $1/6$ of the area of $ABCDEF$, $\triangle ORS$ encloses $1/6$ of the area of the smaller hexagon, and $\triangle ORS$ is equilateral. Let T be the center of $\triangle ORS$. Then triangles TOR, TRS, and TSO are congruent isosceles triangles with largest angle $120°$. Triangle ERS is an isosceles triangle with largest angle $120°$ and a side in common with $\triangle TRS$, so $ORES$ is partitioned into four congruent triangles, exactly three of which form $\triangle ORS$. Since the ratio of the area enclosed by the small regular hexagon to the area of $ABCDEF$ is the same as the ratio of the area enclosed by $\triangle ORS$ to the area enclosed by $ORES$, the ratio is $3/4$.

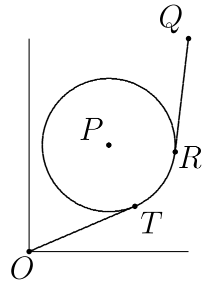


OR

Let M and N denote the midpoints of $\overline{AB}$ and $\overline{AF}$, respectively. Then $MN = AM\sqrt{3}$ since $\triangle AMO$ is a 30°-60°-90° triangle and $MN = MO$. It follows that the hexagons are similar, with similarity ratio $\frac{1}{2}\sqrt{3}$. Thus the desired quotient is $(\frac{1}{2}\sqrt{3})^2 = \frac{3}{4}$.

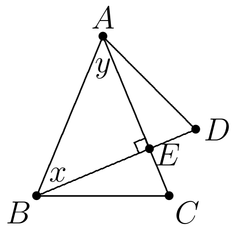


20. **(C)** Let $O = (0, 0), P = (6, 8)$, and $Q = (12, 16)$. As shown in the figure, the shortest route consists of tangent $\overline{OT}$, minor arc TR, and tangent $\overline{RQ}$. Since $OP = 10$, $PT = 5$, and $\angle OTP$ is a right angle, it follows that $\angle OPT = 60°$ and $OT = 5\sqrt{3}$. By similar reasoning, $\angle QPR = 60°$ and $QR = 5\sqrt{3}$. Because O, P, and Q are collinear (why?), $\angle RPT = 60°$, so arc TR is of length $5\pi/3$. Hence the length of the shortest route is $2\left(5\sqrt{3}\right) + \dfrac{5\pi}{3}$.



21. **(D)** Let $\angle ABD = x$ and $\angle BAC = y$. Since the triangles ABC and ABD are isosceles, $\angle C = (180° - y)/2$ and $\angle D = (180° - x)/2$. Then, noting that $x + y = 90°$, we have

$$\angle C + \angle D = (360° - (x + y))/2 = 135°.$$

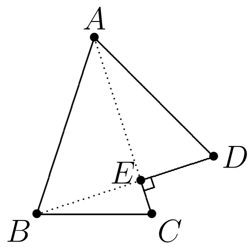


OR

Consider the interior angles of pentagon $ADECB$. Since triangles ABC and ABD are isosceles, $\angle C = \angle B$ and $\angle D = \angle A$. Since $\overline{BD} \perp \overline{AC}$, the interior angle at E measures $270°$. Since $540°$ is the sum of the interior angles of any pentagon,

$$\angle A + \angle B + \angle C + \angle D + \angle E$$
$$= 2\angle C + 2\angle D + 270° = 540°,$$

from which it follows that $\angle C + \angle D = 135°$.



22. **(B)** Because all quadruples are equally likely, we need only examine the six clockwise orderings of the points: $ACBD, ADBC, ABCD, ADCB, ABDC$, and $ACDB$. Only the first two of these equally likely orderings satisfy the intersection condition, so the probability is $2/6 = 1/3$.

23. **(B)** Let a, b, and c be the dimensions of the box. It is given that

$$140 = 4a + 4b + 4c \qquad \text{and} \qquad 21 = \sqrt{a^2 + b^2 + c^2},$$

hence

$$35 = a + b + c \qquad (1) \qquad \text{and} \qquad 441 = a^2 + b^2 + c^2 \qquad (2).$$

Square both sides of (1) and combine with (2) to obtain

$$\begin{aligned} 1225 &= (a+b+c)^2 \\ &= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \\ &= 441 + 2ab + 2bc + 2ca. \end{aligned}$$

Thus the surface area of the box is $2ab + 2bc + 2ca = 1225 - 441 = 784$.

24. **(B)** The k^{th} 1 is at position

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

and $\dfrac{49(50)}{2} < 1234 < \dfrac{50(51)}{2}$, so there are 49 1's among the first 1234 terms. All the other terms are 2's, so the sum is $1234(2) - 49 = 2419$.

OR

The sum of all the terms through the occurrence of the k^{th} 1 is

$$\begin{aligned} & 1 + (2+1) + (2+2+1) + \cdots + (\underbrace{2 + 2 + \cdots + 2}_{k-1} + 1) \\ = \;& 1 + 3 + 5 + \cdots + (2k-1) \\ = \;& k^2. \end{aligned}$$

The k^{th} 1 is at position

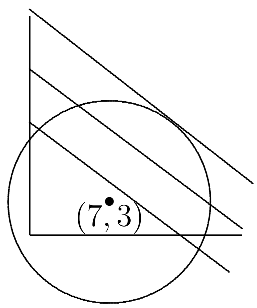
$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}.$$

It follows that the last 1 among the first 1234 terms of the sequence occurs at position 1225 for $k = 49$. Thus, the sum of the first 1225 terms is $49^2 = 2401$, and the sum of the next nine terms, all of which are 2's, is 18, for a total of $2401 + 18 = 2419$.

25. **(B)** The equation $x^2 + y^2 = 14x + 6y + 6$ can be written

$$(x-7)^2 + (y-3)^2 = 8^2,$$

which defines a circle of radius 8 centered at $(7,3)$. If k is a possible value of $3x+4y$ for (x,y) on the circle, then the line $3x+4y=k$ must intersect the circle in at least one point. The largest value of k occurs when the line is tangent to the circle, and is therefore perpendicular to the radius at the point of tangency. Because the slope of the tangent line is $-3/4$, the slope of the radius is $4/3$. It follows that the point on the circle that yields the maximum value of $3x+4y$ is one of the two points of tangency,



$$x = 7 + \frac{3\cdot 8}{5} = \frac{59}{5}, \qquad y = 3 + \frac{4\cdot 8}{5} = \frac{47}{5},$$

or

$$x = 7 - \frac{3\cdot 8}{5} = \frac{11}{5}, \qquad y = 3 - \frac{4\cdot 8}{5} = -\frac{17}{5}.$$

The first point of tangency gives

$$3x + 4y = 3\cdot\frac{59}{5} + 4\cdot\frac{47}{5} = \frac{177}{5} + \frac{188}{5} = 73,$$

and the second one gives $3x+4y = \frac{33}{5} - \frac{68}{5} = -7$. Thus 73 is the desired maximum, while -7 is the minimum.

OR

Suppose that $k = 3x+4y$ is a possible value. Substituting $y = (k-3x)/4$ into $x^2+y^2 = 14x+6y+6$, we get $16x^2 + (k-3x)^2 = 224x + 24(k-3x) + 96$, which simplifies to

$$25x^2 - 2(3k+76)x + (k^2 - 24k - 96) = 0. \qquad (1)$$

If the line $3x+4y=k$ intersects the given circle, the discriminant of (1) must be nonnegative. Thus we get $(3k+76)^2 - 25(k^2-24k-96) \geq 0$, which simplifies to

$$(k-73)(k+7) \leq 0.$$

Hence $-7 \leq k \leq 73$.

26. **(B)** The hypothesis of equally likely events can be expressed as

$$\frac{\binom{r}{4}}{\binom{n}{4}} = \frac{\binom{r}{3}\binom{w}{1}}{\binom{n}{4}} = \frac{\binom{r}{2}\binom{w}{1}\binom{b}{1}}{\binom{n}{4}} = \frac{\binom{r}{1}\binom{w}{1}\binom{b}{1}\binom{g}{1}}{\binom{n}{4}}$$

where r, w, b, and g denote the number of red, white, blue, and green marbles, respectively, and $n = r + w + b + g$. Eliminating common terms and solving for r in terms of w, b, and g, we get

$$r - 3 = 4w, \qquad r - 2 = 3b, \quad \text{and} \quad r - 1 = 2g.$$

The smallest r for which w, b, and g are all positive integers is $r = 11$, with corresponding values $w = 2$, $b = 3$, and $g = 5$. So the smallest total number of marbles is $11 + 2 + 3 + 5 = 21$.

27. **(D)** From the description of the first ball we find that $z \geq 9/2$, and from that of the second, $z \leq 11/2$. Because z must be an integer, the only possible lattice points in the intersection are of the form $(x, y, 5)$. Substitute $z = 5$ into the inequalities defining the balls:

$$x^2 + y^2 + \left(z - \frac{21}{2}\right)^2 \leq 6^2 \qquad \text{and} \qquad x^2 + y^2 + (z-1)^2 \leq \left(\frac{9}{2}\right)^2.$$

These yield

$$x^2 + y^2 + \left(-\frac{11}{2}\right)^2 \leq 6^2 \quad \text{and} \quad x^2 + y^2 + (4)^2 \leq \left(\frac{9}{2}\right)^2,$$

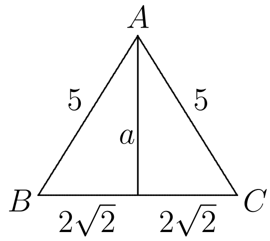
which reduce to

$$x^2 + y^2 \leq \frac{23}{4} \quad \text{and} \quad x^2 + y^2 \leq \frac{17}{4}.$$

If $(x, y, 5)$ satisfies the second inequality, then it must satisfy the first one. The only remaining task is to count the lattice points that satisfy the second inequality. There are 13:

$$\begin{array}{ccccc} (-2,0,5), & (2,0,5), & (0,-2,5), & (0,2,5), & (-1,-1,5), \\ (1,-1,5), & (-1,1,5), & (1,1,5), & (-1,0,5), & (1,0,5), \\ (0,-1,5), & (0,1,5), & \text{and} & (0,0,5). & \end{array}$$

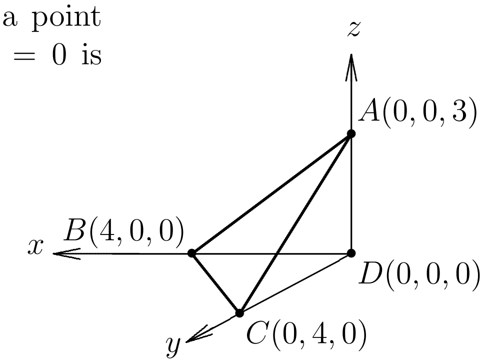
28. **(C)** Let h be the required distance. Find the volume of pyramid $ABCD$ as a third of the area of a triangular base times the altitude to that base in two different ways, and equate these volumes. Use the altitude $\overline{AD}$ to $\triangle BCD$ to find that the volume is 8. Next, note that h is the length of the altitude of the pyramid from D to $\triangle ABC$. Since the sides of $\triangle ABC$ are 5, 5, and $4\sqrt{2}$, by the Pythagorean Theorem the altitude to the side of length $4\sqrt{2}$ is $a = \sqrt{17}$. Thus, the area of $\triangle ABC$ is $2\sqrt{34}$, and the volume of the pyramid is $2\sqrt{34}h/3$. Equating the volumes yields



$$2\sqrt{34}h/3 = 8, \quad \text{and thus} \quad h = 12/\sqrt{34} \approx 2.1.$$

OR

Imagine the parallelepiped embedded in a coordinate system as shown in the diagram. The equation for the plane (in intercept form) is $\frac{x}{4} + \frac{y}{4} + \frac{z}{3} = 1$. Thus, it can be expressed as $3x + 3y + 4z - 12 = 0$. The formula for the distance d from a point (a, b, c) to the plane $Rx + Sy + Tz + U = 0$ is given by



$$d = \frac{|Ra + Sb + Tc + U|}{\sqrt{R^2 + S^2 + T^2}},$$

which in this case is

$$\frac{|-12|}{\sqrt{3^2 + 3^2 + 4^2}} = \frac{12}{\sqrt{34}} \approx 2.1.$$

29. **(C)** Let $2^{e_1}3^{e_2}5^{e_3}\cdots$ be the prime factorization of n. Then the number of positive divisors of n is $(e_1+1)(e_2+1)(e_3+1)\cdots$. In view of the given information, we have

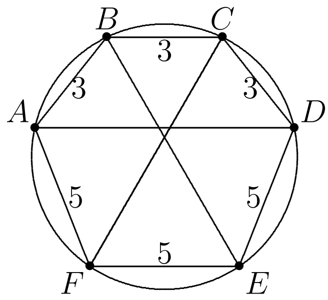
$$28 = (e_1 + 2)(e_2 + 1)P$$

and

$$30 = (e_1 + 1)(e_2 + 2)P,$$

where $P = (e_3+1)(e_4+1)\cdots$. Subtracting the first equation from the second, we obtain $2 = (e_1 - e_2)P$, so either $e_1 - e_2 = 1$ and $P = 2$, or $e_1 - e_2 = 2$ and $P = 1$. The first case yields $14 = (e_1 + 2)e_1$ and $(e_1 + 1)^2 = 15$; since e_1 is a nonnegative integer, this is impossible. In the second case, $e_2 = e_1 - 2$ and $30 = (e_1 + 1)e_1$, from which we find $e_1 = 5$ and $e_2 = 3$. Thus $n = 2^5 3^3$, so $6n = 2^6 3^4$ has $(6 + 1)(4 + 1) = 35$ positive divisors.

30. **(E)** In hexagon $ABCDEF$, let $AB = BC = CD = 3$ and let $DE = EF = FA = 5$. Since arc BAF is one third of the circumference of the circle, it follows that $\angle BCF = \angle BEF = 60°$. Similarly, $\angle CBE = \angle CFE = 60°$. Let P be the intersection of $\overline{BE}$ and $\overline{CF}$, Q that of $\overline{BE}$ and $\overline{AD}$, and R that of $\overline{CF}$ and $\overline{AD}$. Triangles EFP and BCP are equilateral, and by symmetry, triangle PQR is isosceles and thus also equilateral. Furthermore, $\angle BAD$ and $\angle BED$ subtend the same arc, as do $\angle ABE$ and $\angle ADE$. Hence triangles ABQ and EDQ are similar. Therefore,



$$\frac{AQ}{EQ} = \frac{BQ}{DQ} = \frac{AB}{ED} = \frac{3}{5}.$$

It follows that

$$\frac{\frac{AD-PQ}{2}}{PQ+5} = \frac{3}{5} \quad \text{and} \quad \frac{3-PQ}{\frac{AD+PQ}{2}} = \frac{3}{5}.$$

Solving the two equations simultaneously yields $AD = 360/49$, so $m+n = 409$.

OR

In hexagon $ABCDEF$, let $AB = BC = CD = a$ and let $DE = EF = FA = b$. Let O denote the center of the circle, and let r denote the radius. Since the arc BAF is one-third of the circle, it follows that $\angle BAF = \angle FOB = 120°$. By using the Law of Cosines to compute BF two ways, we have $a^2+ab+b^2 = 3r^2$. Let $\angle AOB = 2\theta$. Then $a = 2r\sin\theta$, and

$$\begin{aligned} AD &= 2r\sin(3\theta) \\ &= 2r\sin\theta\cdot(3-4\sin^2\theta) \\ &= a\left(3-\frac{a^2}{r^2}\right) \\ &= 3a\left(1-\frac{a^2}{a^2+ab+b^2}\right) \\ &= \frac{3ab(a+b)}{a^2+ab+b^2}. \end{aligned}$$

Substituting $a = 3$ and $b = 5$, we get $AD = 360/49$, so $m+n = 409$.

OR

In hexagon $ABCDEF$, let $AB = BC = CD = 3$ and minor arcs AB, BC, and CD each be x°, let $DE = EF = FA = 5$ and minor arcs DE, EF, and FA each be y°. Then

$$3x^\circ + 3y^\circ = 360^\circ, \quad \text{so} \quad x^\circ + y^\circ = 120^\circ.$$

Therefore, $\angle BAF = 120^\circ$, so $BF^2 = 3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cos 120^\circ = 49$ by the Law of Cosines, so $BF = 7$. Similarly, $CE = 7$. Using Ptolemy's Theorem in quadrilateral $BCEF$, we have

$$BE \cdot CF = CF^2 = 15 + 49 = 64 \quad \text{so} \quad CF = 8.$$

Using Ptolemy's Theorem in quadrilateral $ABCF$, we find $AC = 39/7$. Finally, using Ptolemy's Theorem in quadrilateral $ABCD$, we have $AC^2 = 3(AD) + 9$ and, since $AC = 39/7$, we have $AD = 360/49$ and $m + n = 409$.

Note. *Ptolemy's Theorem:* If a quadrilateral is inscribed in a circle, the product of the diagonals equals the sum of the products of the opposite sides.