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Fusing Dots, Antidots, and Black Holes

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Outline of Talk

1. Counting on Mars. Three fingers on one hand and four on the other. Or 3 1/2 on each.

2. Representation machines.

3. Antidots and subtraction.

4. Black holes and Fibonacci

5. The base 3/2 machine.

6. The base -4 machine.

7. Puppies, mice and kittens.

Imagine that you live on another planet where you and everyone else has just three and a half fingers on each hand. You might use just 7 digits and powers of 7 when counting: $1, 2, 3, 4, 5, 6, 10, 11, 12 \ldots$. You'd be counting in what is called base-7. We're going to explore the arithmetic of base 7 and also several other unusual bases.

1 Representations

Many thanks to Jim Tanton of St Marks School for the idea of exploding dots. We're going to explore several machines that enable us to represent positive integers and some other real numbers is some odd ways. Initially, we're given a two-way infinite tape with empty squares, with a heavy line (a bar) at one place on the tape: 
To represent a number n, we put n dots in the square just to the left of the bar, and let the machine go to work. This square, also called a box, is called the *unit box*.

2 Introduction

This paper discusses several methods of representing numbers, and several ways to understand these methods of representation. We begin with what is called *decimal representation*, the ordinary method we use to represent integers and fractions. Because the method of representation is an important starting point in learning the arithmetic of integers and decimals, we shall explore alternative methods of representation, that is, representation using bases other than our usual base 10 ten. This is roughly akin to the idea that one does not really understand one's own language until we learn a second language. Instead of trying to develop representation in an arbitrary base b, we select a specific base for the sake of clarity. This is base 5 representation. Later we will discuss other representation including those for which the base b is not a positive integer. We also explore the system of enumeration when b is a rational number but not an integer, and then when b is a negative integer. Finally, we'll also see that it is possible for b to be irrational.

3 Place Value Representation

The *place value* interpretation of 4273 is $4000+200+70+3$, which is a *sum* of *multiples* of *powers* of 10. The relevant powers of 10 are $10^3 = 1000, 10^2 = 100, 10^1 = 10$, and $10^0 = 1$. Each one has a *coefficient* or multiplier, $4, 2, 7$, and 3, respectively. Thus $4 \cdot 10^3, 2 \cdot 10^2, 7 \cdot 10^1$, and $3 \cdot 10^0$ are **multiples** of **powers** of 10 and therefore 4273 is a **sum** of **multiples** of **powers** of 10. So place value notation means decimal notation in this case. Each of the addends in the expanded form of a number will be called a single-place number. Thus for example $4 \cdot 10^3$ is a single place number.

Once we learn how to do arithmetic with single place numbers, we can use that knowledge along with the *distribution property* of multiplication over addition, to do arithmetic with decimal numbers in general. This represents a key virtue of place value: it enables arithmetic computation. The fact that the basic arithmetic operations can be efficiently performed by effectively teachable algorithms was the reason that the place value system, which was only introduced into Europe in the late middle ages (around 1200), supplanted the well entrenched system of Roman numerals. Here is an example. Find the product $23 \cdot 41$. First recognize each of these numbers as a place value number, $23 = 20 + 3$ and $41 = 40 + 1$. Then

$$\begin{aligned}
23 \cdot 41 &= (20+3)\cdot(40+1) \\
&\overset{1}{=} (20+3)40 + (20+3)1 \\
&\overset{2}{=} 20\cdot 40 + 3\cdot 40 + 20\cdot 1 + 3\cdot 1 \\
&\overset{3}{=} 2\cdot 10\cdot 4\cdot 10 + 3\cdot 4\cdot 10 + 2\cdot 10\cdot 1 + 3\cdot 1 \\
&\overset{4}{=} 8\cdot 10^2 + 12\cdot 10 + 2\cdot 10 + 3\cdot 10^0 \\
&\overset{5}{=} 8\cdot 10^2 + (10+2)\cdot 10 + 2\cdot 10 + 3\cdot 10^0 \\
&\overset{6}{=} 9\cdot 10^2 + 1\cdot 10^2 + 2\cdot 10 + 3\cdot 10^0 \\
&\overset{7}{=} 9\cdot 10^2 + 4\cdot 10 + 3\cdot 10^0 \\
&\overset{8}{=} 943,
\end{aligned}$$

where, we have used the distribution property of multiplication over addition in 1, 2 and 6; commutativity of multiplication and addition in 3 and 6; and place value notation in 6,7, and 8. Of course we have also used the digit multiplication table 4 and the digit addition table in 7.

Another objective here is to establish methods of translating between decimal representations and base b representations. In other words, we are given a number expresses as a sum of multiples of powers of 10 and wish to rewrite the number as a **sums of multiples of powers** of b, where b is an integer bigger than 1. For convenience, let us assume for sections 2, 3, and 4 that $b = 5$. The same procedures work no matter what the value of b is, but fixing the value of b here makes discussion much easier. Of course there is also the problem of translating from a base b representation into a decimal representation, and this process is called *interpretation*.

The notation 2113_5 is interpreted as a **sum** of **multiples** of **powers** of 5, just as the decimal number 4273 was above. The *subscript* 5 must be attached unless we are using base 10, because 10 is the default value of the base. Thus $2113_5 = 2 \cdot 5^3 + 1 \cdot 5^2 + 1 \cdot 5^1 + 3 \cdot 5^0 = 250 + 25 + 5 + 3 = 283$. The process of finding the decimal (ie, base 10) value of a number from its base 5 representation is called *interpreting*. Thus we interpreted 2113_5 as 283. The reverse process, that of finding the base 5 representation of an integer expressed in decimal notation is harder and more interesting. There are two methods, (a) *repeated subtraction* and (b) *repeated division*. Each method has some advantages over the other.

4 Repeated Subtraction

To see how to use repeated subtraction, first make a list of all the integer powers of 5 that are not bigger than the number we are given. In the case of 283, we need the powers $5^0 = 1, 5^1 = 5, 5^2 = 25$, and $5^3 = 125$. Next repeatedly subtract the largest power of 5 that is less than or equal to the *current number* (which changes during the process). So we have $283 = 125 + 158$. At this point our current number becomes 158 and we repeat the process. Then $283 = 125 + 158 = 125 + 125 + 33$, and our current number is 33. Repeating the process on 33 gives $33 = 25 + 8$ and incorporating that in the above gives $283 = 2 \cdot 125 + 25 + 8 = 2 \cdot 125 + 1 \cdot 25 + 8$. Continuing this with 8 leads to $283 = 2 \cdot 5^3 + 1 \cdot 5^2 + 1 \cdot 5^1 + 3 \cdot 5^0$, which is a sum of multiples of powers of 5, just what we want. Thus $283 = 2113_5$, just as we saw above. Repeated subtraction has two advantages over the repeated division method. First, it is closely related to the definition, hence it leads to a better conceptualization. Second, it can be used in other situations when repeated division cannot, as in the case of Fibonacci representation.

5 Repeated Division

The repeated division method requires that we repeatedly divide the given integer by base 5 and record the remainder at each stage. First we divide 283 by 5 to get $283 \div 5 = 56.6$. We can interpret this as $283 = 5 \cdot 56 + 3$, so the *quotient* is 56 and the *remainder* is 3. Notice that the remainder can never exceed 5 since in such a case the quotient would have been larger. Next divide the quotient by 5 and record the new quotient and the remainder. Thus $56 = 5 \cdot 11 + 1$. Repeat the process with the new quotient $11 = 5 \cdot 2 + 1$ and finally, $2 = 5 \cdot 0 + 2$. Next write the remainders in reverse order, $2, 1, 1$, and 3 to get 2113_5 as the base 5 representation of 283. You'll see why the order must be reversed in the following example.

Example 1. Repeated Division To see why $283 = 2113_5$, we can repeatedly replace each quotient with its value obtained during the division process. Thus

$$
\begin{aligned}
283 &= 5 \cdot 56 + 3 \\
&= 5(5 \cdot 11 + 1) + 3 \\
&= 5(5(5 \cdot 2 + 1) + 1) + 3 \\
&= 5(5 \cdot 5 \cdot 2 + 5 \cdot 1 + 1) + 3 \\
&= 5 \cdot 5 \cdot 5 \cdot 2 + 5 \cdot 5 \cdot 1 + 5 \cdot 1 + 3 \\
&= 2 \cdot 5^3 + 1 \cdot 5^2 + 1 \cdot 5^1 + 3 \cdot 5^0 \\
&= 2113_5
\end{aligned}
$$

The advantage of repeated division is that it is computationally more efficient. Also, the method of justification can be applied in other situations (synthetic division and Euclidean algorithm). When we get to the section on fusing dots, you'll see in yet another way why it makes sense to record the remainders upon division by b.

6 Repeated Multiplication and Repeated Subtraction

In sections 3 and 4 we saw two methods (algorithms) for writing a given integer in a base different from 10. Before we consider representing fractions, let's review the place value ideas in decimal notation. For example, 5.234 is, as in the first part, a **sum of multiples of powers of** 10. This time, the powers are (except one) negative exponents:

$$5.234 = 5 \cdot 10^0 + 2 \cdot 10^{-1} + 3 \cdot 10^{-2} + 4 \cdot 10^{-3}.$$

Using this interpretation as a guide, we can interpret 0.124_5 similarly, as a sum of multiples of (negative) powers of 5. Thus

$$\begin{aligned} 0.124_5 &= 1 \cdot 5^{-1} + 2 \cdot 5^{-2} + 4 \cdot 5^{-3} \\ &= \frac{1}{5} + \frac{2}{25} + \frac{4}{125} = \frac{25 + 10 + 4}{125} \\ &= \frac{39}{125} \end{aligned}$$

As in the discussion of integers, there are two methods for dealing with numbers in the range $0 < x < 1$. They are called (a) *repeated subtraction* and (b) *repeated multiplication*. As before, each has advantages over the other.

Example 2. Repeated Subtraction To use the method of repeated subtraction on 39/125, first list the powers of 5 with negative integer exponents:

$$5^{-1} = 1/5,\ 5^{-2} = 1/25,\ 5^{-3} = 1/125, \ldots.$$

Find the largest of these powers of 5 and subtract it from the original number. Thus $39/125 - 1/5 = 14/125$. Therefore, $39/125 = 1/5 + 14/125$. Now repeat the process on the number 14/125. Note that $1/25 = 5/125$. Thus, $14/125 - 1/25 = 8/125$. Therefore, $14/125 = 1/25 + 9/125$. Putting this together with the arithmetic above, we have

$$\frac{39}{125} = \frac{1}{5} + \frac{1}{25} + \frac{1}{25} + \frac{4}{125}.$$

Again dealing with the extra part, $4/125 - 1/125 = 3/125$, etc . At this point we can anticipate the final arithmetic:

$$\begin{aligned} \frac{39}{125} &= \frac{1}{5} + \frac{1}{25} + \frac{1}{25} + \frac{1}{125} + \frac{1}{125} + \frac{1}{125} + \frac{1}{125} \\ &= 1 \cdot 5^{-1} + 2 \cdot 5^{-2} + 4 \cdot 5^{-3} \\ &= 0.124_5 \end{aligned}$$

The method of repeated multiplication is much quicker and does not require so much fraction arithmetic.

Example 3. Repeated Multiplication To find the base 5 representation of 13/54, we repeatedly multiply by 5. Following each multiplication by 5, split the result into its integer part and its fractional part:

$$\frac{39}{125} \cdot 5 = \frac{39 \cdot 5}{25 \cdot 5} = \frac{39}{25} = 1 + \frac{14}{25}.$$

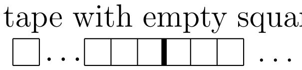
Each integer part is a digit in the representation. Thus $39/125 = 0.1\ldots_5$. Now repeat the process using the new fractional part, 14/25:

$$\frac{14}{25} \cdot 5 = \frac{14}{5} = 2 + \frac{4}{5}.$$

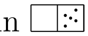
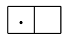
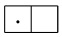
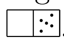
Thus $39/125 = 0.12\ldots_5$. Repeating the process, $\frac{4}{5} \cdot 5 = 4 + 0$. Since the fractional part is 0, we are done (why?). Thus, $\frac{39}{125} = 0.124_5$.

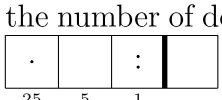
Of course, not all rational numbers have base 5 representations that terminate (ie, end in all 0's from some point on). But there is an easy way to tell, and a great notation to use when the representation does not terminate. Consider the problem of finding the *binary* (that is, base 2) representation of $\frac{1}{3}$. Using repeated multiplication, we get $\frac{1}{3} \cdot 2 = 0 + \frac{2}{3}$. Then $\frac{2}{3} \cdot 2 = 1 + \frac{1}{3}$. Thus we see the same fractional part $\frac{1}{3}$ occurs again. The first two digits are 0 and 1, so we have $\frac{1}{3} = 0.01\ldots_2$, but we can see that the block 01 continues to recur. The slick way to write this number $0.01010101\ldots$ is $0.\overline{01}_2$. When the representation repeats in blocks, the number can be regarded as the sum of an infinite geometric series. In this case it is $2^{-2} + 2^{-4} + 2^{-6} + \cdots$. There is a formula for finding the sum of the geometric series $a + ar + ar^2 + ar^3 + \cdots$. It is $\frac{a}{1-r}$, and this holds whenever $|r| < 1$. Thus $2^{-2} + 2^{-4} + 2^{-6} + \cdots = \frac{2^{-2}}{1-2^{-2}} = \frac{1/4}{3/4} = \frac{1}{3}$, just as we knew.

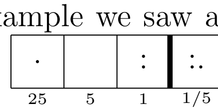
7 Fusing Dots

Many thanks to Jim Tanton of St Marks School for the idea of exploding dots. We're going to explore several machines that enable us to represent positive integers and some other real numbers is some odd ways. Initially, we're given a two-way infinite tape with empty squares, with a heavy line (a bar) at one place on the tape: 

To represent a number n, we put n dots in the square just to the left of the bar, and let the machine go to work. This square, also called a box, is called the *unit box*.

1. The $\boxed{1 \leftrightarrow 5}$ machine. In this machine, whenever five dots occupy the same square, they are erased (they 'fuse') and they are replaced with one dot in the square to their left. Thus the five dots in  fuse to become one dot in . There will also be times when we need to reverse the process in which case one dot in a square is replaced by 5 dots in the square to the right. We'll call this process *explosion*. Thus  explodes to become .

   (a) How can we use this machine to represent a positive integer, like 27? What happen when we put 27 dots in the unit box? The answer is that we can assign to each box to the left of the bar a value. The integer is the sum of the products of the values times the number of dots in each box with the given value. For example, 

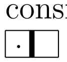
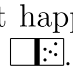
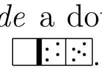
   $\cdots$, has the value $25 + 0 + 2 = 27$. We agree to write this as 102 instead of putting dots in boxes. Here is another example. Go back to the example we saw above, but add a few dots to the right of the bar. 

   $\cdots$, has the value $25 + 0 + 2 + 3/5 = 27.6$. See the exercises that follow. Problem 9 will help you understand base 5 representation.

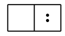
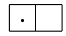
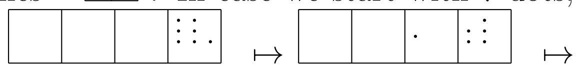
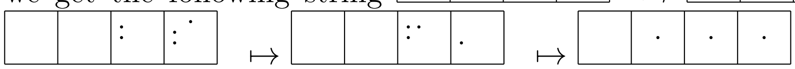
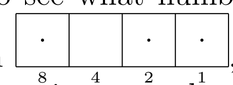
   (b) How can we use this model to add two numbers? Find the values of the numbers represented as 2432 and 2341. Find the sum of 2341 and 2432 using the exploding dot model.

   (c) How can we use this model to subtract two numbers? In particular, work out $2432 - 2341$.

   (d) How can we use this model to multiply two numbers?

   (e) How can we use this model to understand fractions? For example, consider what happens when we *explode* a dot repeatedly. Thus  becomes . And this becomes . So, we have $1 = 0.5 = 0.45 = 0.445 = 0.4445 = \ldots = 0.\overline{4}$. In fact, this infinite geometric series converges, as we know.

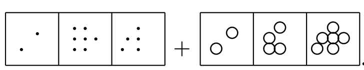
2. The $\boxed{1 \leftrightarrow 2}$ machine. In this machine, whenever two dots occupy the same square, they are fused together to be one dot in the square to

their left. Thus  becomes . In case we start with 7 dots, we get the following string  . Instead of constructing a string of squares and dots, we call this representation 111. As an exercise, see what you get for 19 dots. Also, check to see if the order in which the explosions take place affects the final distribution of dots. Since each dot in a square is worth two dots in the square to its right, we can assign values to each square to see what number is represented. For example, the dot configuration , has the value $8 + 2 + 1 = 11$. Of course it is not a surprise to us that this is just binary representation. Problem 8 is about the machine $\boxed{1 \leftrightarrow 2}$.

3. The $\boxed{1 \leftrightarrow 10}$ machine. In this machine, whenever ten dots occupy the same square, they fuse together as one dot in the square to their left.

   Exercise. Use the $\boxed{1 \leftrightarrow 10}$ machine to find the decimal representation of 275.

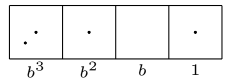
   Let's examine subtraction with the $\boxed{1 \leftrightarrow 10}$ machine. Consider the problem $275 - 246$. To accomplish this we first devise a notion of negation using *antidots*. We allow two types of symbols in squares, dots $\cdot$ and antidots $\circ$. They annihilate each other. Thus, we have

   . Can you finish the job? Next using the same machine, examine what happens when a single dot in the units position repeatedly explodes producing 10 new dots for each explosion. Imagine what happens if this process is repeated infinitely many times.

4. The $\boxed{2 \leftrightarrow 3}$ machine. In this machine, whenever three dots occupy the same square, they fuse together to become two dots in the square to their left. Let's work out the notation for each of the numbers from 1 to 15.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$R(n)$	1	2	20	21	22	210	211	212	2100	2101	2102	2120	2121	2122	21010

Work your way up to the representation for 24. Notice that the number of digits in the representation jumps as we move to $3, 6, 9$ and 15.

Where is the next jump. Why? Is this machine a base-b representation machine for some number b? If so, then  would have the value $2b^3 + b^2 + 1$. Compute the value of the representation 2101 without help from the chart above.

Realizing that each pair of dots in a box is worth three in the next box, we can derive the equations $2b = 3, 2b^2 = 3b, 2b^3 = 3b^2$, etc, all of which give us $b = 3/2$.

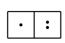
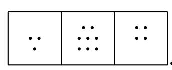
(a) Find the representation of 123 for this machine.

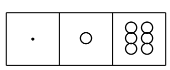
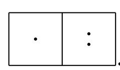
(b) Find necessary and sufficient conditions on a digit string for it to represent an integer.

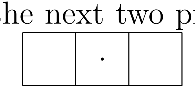
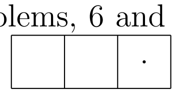
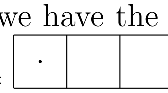
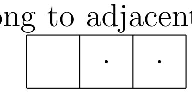
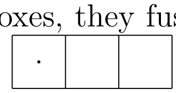
5. The $\boxed{1 \leftrightarrow x}$ machine. In this machine, whenever x dots occupy the same square, they fuse and they are replaced with one dot in the square to their left. This leads to polynomial arithmetic. Let's work out an example of polynomial division using the $\boxed{1 \leftrightarrow x}$ machine.

   (a) Represent $3x^2 + 8x + 4$ in the $\boxed{1 \leftrightarrow x}$ machine.

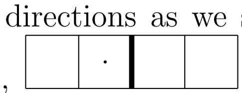
   (b) Represent $x + 2$ in the $\boxed{1 \leftrightarrow x}$ machine.

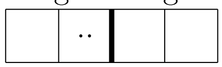
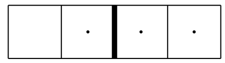
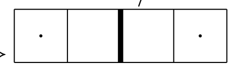
   (c) Now find all instances of  in .

   (d) Next try  $\div$ .

For each of the next two problems, 6 and 7, we have the same 'fusion scheme':  +  = . In other words, when dots belong to adjacent boxes, they fuse to give a dot in the next box over:  = 

Let's call this the $\boxed{1 \leftrightarrow 1, 1}$ machine. In the final representations, we are not allowed to have more than one dot in a box.

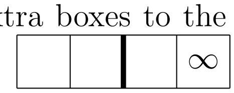
6. In the first part, we also need a two-way infinite row of boxes. You'll see why we need both directions as we start to count. Of course, 1 is represented as usual,  The bold vertical segment represents a special location, which for base b, we call a *radix point*.

Instead of using dots in boxes, its more convenient from here on to express integers as digit strings. So 2 is $2 \Rightarrow 1.11 \Rightarrow 10.01$. The box/dot diagram here is  $\Rightarrow$  $\Rightarrow$ 
In words, one of the two dots in the first box exploded producing dots in the two previous boxes. So $2 = 10.01$. Then $3 = 11.01 \Rightarrow 100.01, 4 = 101.01$. Now 5 is tricky: $5 = 4 + 1 = 101.01 + 1 = 102.01 \Rightarrow 101.12 \Rightarrow 101.1111 \Rightarrow 110.0111 \Rightarrow 1000.1001$. Then $6 = 1001.1001 \Rightarrow 1010.0001$.

(a) Is this a base system in the usual sense. In particular, is there a real number b for which

$$6 = b^3 + b + b^{-4}?$$

(b) Find the representations of the next 5 integers, $7, 8, 9, 10$, and 11.

7. Here's the $\boxed{1 \leftrightarrow 1, 1}$ machine with a *black hole*. It only takes left infinite strings of boxes with two extra boxes to the right of the radix point, one of which is a black hole:  Here's how this works. Any dots in the box marked ∞ at the end disappear. Otherwise it works just like the machine in question 6. So, $1 = 1$, $2 = 2.00 \Rightarrow 1.11 \Rightarrow 10.01 \Rightarrow 10.00 = 10$, $3 = 2 + 1 = 10 + 1 = 11 = 100$, $4 = 12.00 \Rightarrow 11.11 \Rightarrow 100.11 \Rightarrow 101.00$. To find the representation of 5, start with the representation 4, and add one: $5 = 102.00 \Rightarrow 101.11 \Rightarrow 110.01 \Rightarrow 1000.01 \Rightarrow 1000.00 = 1000$.

   (a) Find the representations of the next 5 integers, $7, 8, 9, 10$, and 11.

   (b) Is this a base system in the usual sense. For example, notice that $4 = 101$ in the machine, so we might expect that there a number b for which

$$4 = b^2 + 1?$$

   (c) Prove that no positive integer representation has a 1 in any position to the right of the decimal.

8. See the worksheet problems.

9. Build the $\boxed{1 \leftrightarrow 5}$ and $\boxed{2 \leftrightarrow 5}$ machine representations for each of the numbers $A = 737$ and $B = 831$.

(a) Add the two numbers in the $\boxed{1 \leftrightarrow 5}$ framework and then find the decimal value of the sum. Since $A + B = 1568$, you can check to see if your answer is correct.

(b) Subtract the smaller from the larger in the $\boxed{1 \leftrightarrow 5}$ framework. Since $B - A = 94$, you can check to see if your answer is correct.

(c) Multiply the two numbers in the $\boxed{1 \leftrightarrow 5}$ in the usual way by constructing the digit product and digit sum tables first as you might for decimal arithmetic. Since $A \cdot B = 612447$, you can check to see if your answer is correct.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

(d) Add the two numbers in the $\boxed{2 \leftrightarrow 5}$ framework.

(e) Subtract the smaller from the larger in the $\boxed{2 \leftrightarrow 5}$ framework.

(f) Multiply the two numbers in the $\boxed{2 \leftrightarrow 5}$ framework in the usual way by constructing the digit product and digit sum tables first as you might for decimal arithmetic.

10. Next consider the machine $\boxed{1 \leftrightarrow n}$ in which the number of dots that get fused together depends upon which box is being considered. In the units box, it works like the $\boxed{1 \leftrightarrow 2}$ machine, and in the next box to the left, it works like the $\boxed{1 \leftrightarrow 3}$ machine. For example, $0009 \to 0041 \to 0111$, which means that if we started with 9 dots in the units positions, four fusions would occur leaving one dot in the units box and 4 in the next box, after which there would be one fusion of 3 dots, leading to

	·	·	·

which we write as 0111.

11. Find the representation of 1000 in the $\boxed{1 \leftrightarrow n}$ machine described above.

12. Find the representation of $7 \cdot 7! + 6 \cdot 6! + 5 \cdot 5! + 4 \cdot 4! + 3 \cdot 3! + 2 \cdot 2! + 1 \cdot 1!$ in the $\boxed{1 \leftrightarrow n}$ machine described above.

13. This was problem 25 on the 1999 AHSME. There are unique integers $a_2, a_3, a_4, a_5, a_6, a_7$ such that

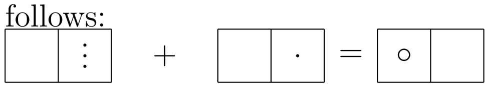
$$\frac{5}{7} = \frac{a_2}{2!} + \frac{a_3}{3!} + \frac{a_4}{4!} + \frac{a_5}{5!} + \frac{a_6}{6!} + \frac{a_7}{7!},$$

where $0 \le a_i < i$ for $i = 2, 3, \ldots, 7$. Find a_2, a_3, a_4, a_5, a_6, and a_7.

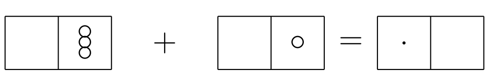
8 Place Value with Negative Bases

In this section we study the consequences of using a negative base for arithmetic. As we did in the previous section, we'll pick a sample base and stick with it throughout. You'll see easily how to modify the ideas when other bases are used. We'll pick negative 4 as our base. Here we allow ourselves the digits $0, 1, 2, 3$. Let us first **interpret** a number written in base -4. For example take 113.3_{-4}. We interpret this as a sum of multiples of powers of -4: $1 \cdot (-4)^2 + 1 \cdot (-4)^1 + 3 \cdot (-4)^0 + 3 \cdot (-4)^{-1} = 16 - 4 + 2 - 3/4 = 13.25$. Thus, we write $13.25 = 113.3_{-4}$. The methods for finding the base negative four representation of a positive integer are interesting. Also of interest are methods for finding the base -4 representation of rational numbers r satisfying $0 < r < 1$. We can find the base -4 representation of 13.25 by combining these two methods.

The machine we can use is denoted $\boxed{-1 \leftarrow 4}$. Its called this because when four dots accumulate in a box, they fuse, causing an anti-dot to be formed in the next box to the left. Similarly when four antidots accumulate, they fuse to give a dot in the box to the left. This machine can be drawn as follows:

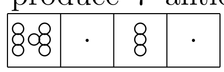
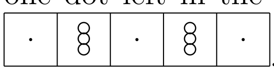
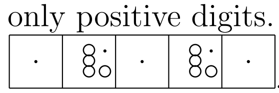
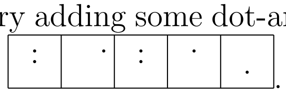


and



Example 1. Repeated Division To see why $477 = 21211_{-4}$, we can repeatedly replace each quotient with its value obtained during the division process. Its important to remember that the remainders cannot be negative numbers. Thus

$$\begin{aligned}
477 &= -4 \cdot -119 + 1 \\
&= -4(-4 \cdot 30 + 1) + 1 \\
&= -4(-4(-4 \cdot -7 + 2) + 1) + 1 \\
&= -4(-4(-4(-4 \cdot 2) + 1 + 1) + 1 \\
&= -4(-4(-4(-4 \cdot 2 + 1) + 2) + 1) + 1 \\
&= 2(-4)^4 + 1(-4)^3 + 2(-4)^2 + 1(-4)^1 + 1(-4)^0 \\
&= 21211_{-4}
\end{aligned}$$

Here's how the fusing dot machine above would process the number 477. First, there would be 119 fusions that would produce 119 antidots in the second box and one dot in the right box. Then 29 fusions would take place, producing 29 dots in the third box and 3 antidots in the second box. Then 7 fusions would take place to produce 7 antidots in the fourth box with one dot left in the third box:  The final fusion produces: . So, can we say that the base −4 representation of 477 is 1 − 31 − 31? Of course not. We can use only positive digits. So what can we do? Try adding some dot-antidot pairs. . From here its easy: . In other words, 21211_{-4}.

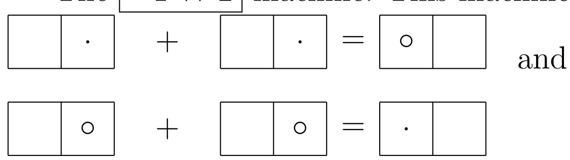
The algorithms for finding the base −4 representation of fractions is even more interesting. My AwesomeMath student Eliot Levmore, suggested the following algorithm, related to repeated multiplication. To find the base −4 representation of 7/20, first note that our number is positive, so it looks like 1.*abcd*.... That means the .*abcd*... has value $7/20 - 1 = -13/20$. Multiply $-13/20$ by -4 to get $52/20 = 13/5 = 3 - 2/5$, so the digit a is 3. Then multiply $-2/5$ by -4 to get $8/5$ which we can write as $2 - 2/5$. Our representation is 1.32*cd*.... Now $-2/5 \cdot -4 = 8/5$ again, and we can see that the digit 2 repeats. Thus $7/20 = 1.3\overline{2}_{-4}$. Can you prove that this is correct? Which rational numbers less than 1 require a digit 1 in the unit's position? Levmore again provides the answer. To see what it is, ask yourself the question, What is the largest rational number representable as $0.x_1x_2\ldots$?

In algebra, you learn a method for converting a repeating decimal to a ratio of two integers. We can do that here also. Let $t = 1.3\overline{2}_{-4}$. Then $16t = 132.\overline{2}$ and $16t - t = 15t = 132.2_{-4} - 1.3_{-4} = 132.3 = 21/4 \div 15 = 7/20$.

This algorithm is not perfect, however because the subtraction idea can lead to digits larger than 3. Can you devise another method that avoids this problem?

Here's an idea. To find the base -4 representation of a fraction, we repeatedly multiply by 16, and produce two digits at a time. This way we can avoid the difficulties posed by the negative numbers.

The $\boxed{-1 \leftrightarrow 2}$ machine. This machine is defined by two equations,



1. Find the $\boxed{-1 \leftrightarrow 2}$ machine representations for the integers from 1 to 10.

2. What number is represented by 110110101?

3. Is this a base b machine like the $\boxed{2 \leftrightarrow 3}$ machine? If so, find b.

4. What is the representation of 63? Can you find an algorithm that works for n without finding representations of all the numbers 1 through $n-1$?

5. What is the representation of 90?

6. Go back to the $\boxed{1 \leftrightarrow 2}$ machine. Find the representation of 1/3.

7. Find the representation of 1/3 in base -2. That is find the representation in the $\boxed{-1 \leftrightarrow 2}$ machine.

9 Fibonacci, Factorial, and Balance-Pan Enumeration

Fibonacci Representation The Fibonacci numbers $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5 \ldots$ are defined so that after the first two, every one is the sum of the last two. In other words, $F_1 = 1, F_2 = 2$ and $F_{n+2} = F_n + F_{n+1}$. Thus the sequence is $1, 2, 3, 5, 8, 13, 21, 34, 55, 89 \ldots$. In the case of Fibonacci representation, we need only two digits, 0 and 1. These represent the absence or presence of the corresponding Fibonacci number. To represent a number in Fibonacci representation, use the method of repeated subtraction.

Example 4. Fibonacci Representation To find the Fibonacci representation of 100, find the largest Fibonacci number less than or equal to 100. Then subtract it and repeat the process. Thus $100 = 89 + 11$. Thus $100 = 89 + 11 = 89 + 8 + 3 = 1000010100_f$. Of course, the 1's tell us which Fibonacci numbers are added, and the 0's tell us to leave out the number: 1000010100_f means $1F_{10}+0F_9+0F_8+0F_7+0F_6+1F_5+0F_4+1F_3+0F_2+0F_1$. Notice that the representation 1000010100_f has at least one 0 between each pair of 1's. Try to figure out why this is always the case this before reading on. We'll return to this representation later. How can we do arithmetic with numbers represented this way? Addition is not very hard. Let's use the addition $87 + 31$.

Example 5. Fibonacci Arithmetic In the notation we (slightly) abuse the notation by using the coefficient 2 at times.

	89	55	34	21	13	8	5	3	2	1
87		1	0	1	0	1	0	1	0	0
+31				1	0	1	0	0	1	0
		1	0	2	0	2	0	1	1	0
		1	0	2	0	2	1	0	0	0
		1	0	2	1	1	0	0	0	0
		1	1	1	0	1	0	0	0	0
118	1	0	0	1	0	1	0	0	0	0

The addition process repeatedly makes use of the fact that the sum of two successive Fibonacci numbers is the next one. In the representation, therefore, you never need to have two successive 1's. Another example might be helpful here. How would you carry out $21 + 21$? That is $1000000_f + 1000000_f = 2000000_f = 1110000_f = 10010000_f = 34 + 8 = 42$ Can you devise an algorithm for multiplication?

Factorial Representation Here the idea is to represent each number as a sum of multiples of factorials. The basic building blocks are the numbers $1 = 1!, 2 = 2!, 6 = 3!, 24, 120, 720, \ldots$. The coefficients allowed for $n!$ are the numbers from 1 up to n. Of course, using $n + 1$ as a coefficient for $n!$ would not be needed since $(n + 1)n! = (n + 1)!$. The table below lists the representations of the first twelve positive integers. How can we find the factorial base for a positive integer N? The answer is by repeated division. First, divide by two and write the remainder in the rightmost position. Of course the remainder is either 1 or 0 depending on the parity of the N. Next divide the quotient by three, and again write down the remainder. Continue this process until the quotient is zero.

Example 6. Repeated Division To find the Factorial representation of $N = 127$ first divide by 2. The first remainder is 1, and the quotient is 63. Dividing 63 by three yields a quotient of 21 and a remainder of 0. Then dividing 21 by four yields quotient 5 with remainder 1. Finally, divide by five to get a quotient of 1 and a remainder of 0. Thus $127 = 10101_!$. That is, $127 = 5! + 3! + 1!$.

Arithmetic in factorial notation is not very hard. Let's pursue addition.

Example 7. Factorial Arithmetic Consider the problem $65 + 21$, which in factorial notation is $2221_! + 311_!$ since $65 = 2 \cdot 4! + 2 \cdot 3! + 2 \cdot 2! + 1 \cdot 1!$ while $21 = 3 \cdot 3! + 1 \cdot 2! + 1 \cdot 1!$. So

$$\begin{array}{r} 2 \cdot 4! + 2 \cdot 3! + 2 \cdot 2! + 1 \cdot 1! \\ + 3 \cdot 3! + 1 \cdot 2! + 1 \cdot 1! \\ \hline 2 \cdot 4! + 5 \cdot 3! + 3 \cdot 2! + 2 \cdot 1! \end{array}$$

But $5 \cdot 3! = (4 + 1) \cdot 3! = 4 \cdot 3! + 3! = 4! + 3!$. So the sum is just $3 \cdot 4! + 2 \cdot 3! + 2! = 3210_!$.

Balance-Pan Enumeration Now let's turn our attention to balance-pan enumeration. Think about how you would arrange four weights on a two-pan balance to weigh out each of the numbers from 1 to 40, and what weights would you use for such a project. Since there are three things you can do with each weight (put it on the left pan, the right pan, or not use it), there are at most $3^4 = 81$ arrangements of the four weights. One of these is to do nothing with each weight, and for every other arrangement, there is the opposite arrangement where each weight on the left pan is moved to the right pan, and vice-versa. So there are just 40 possible values to be weighted with four weights. A little playing with this leads to the possibility of using the first four powers of 3, $3^0 = 1, 3^1 = 3, 3^2 = 9$, and $3^4 = 27$. We can let the digits $0, 1$, and $\overline{1}$ mean a) don't use the weight, b) put the weight on the left pan, and c) put the weight on the right pan. We'll always put the object to be weighed in the right pan, so we will need to arrange the weights so the sum of the weights in the left pan is at least as large as the sum of the weights in the right pan. Now the representation $11\overline{1}0_b = 1 \cdot 3^3 + 1 \cdot 3^2 - 1 \cdot 3^1 + 0 \cdot 3^0 = 27 + 9 - 3 = 33$, for example.

Example 8. Balance Pan Arithmetic

$$\begin{array}{r} 1\ \overline{1}\ 1\ 1 \\ \times\ 1\ 0\ \overline{1} \\ \hline \overline{1}\ 1\ \overline{1}\ \overline{1} \end{array}$$

$$\begin{array}{r} \underline{1\ \bar{1}\ 1\ 1\ 0\ 0} \\ 1\ \bar{1}\ 1\ \bar{1}\ \bar{1}\ \bar{1} \end{array}$$

Of course, this is just a slick way to show that $22 \times 8 = 176$.

To do arithmetic of integers represented in balance-pan representation, we need the addition and multiplication tables for digits, just as we learned in third grade for decimal representation.

$+$	$\bar{1}$	0	1
$\bar{1}$	$\bar{1}1$	$\bar{1}$	0
0	$\bar{1}$	0	1
1	0	1	$1\bar{1}$

$\times$	$\bar{1}$	0	1
$\bar{1}$	1	0	$\bar{1}$
0	0	0	0
1	$\bar{1}$	0	1

Of course, the balance pan representation of a number is closely related to its ternary representation. In fact $M = (a_k a_{k-1} \ldots a_0)_b$ where $a_i \in \{0, \bar{1}, 1\}$ if and only if $M = \sum_{i \in P} 3^i - \sum_{i \in N} 3^i$, where P is the set of indices i for which $a_i = 1$ and N is the set of indices i for which $a_i = \bar{1}$. For yet another example, consider the case $M = 15 = 3^3 - 3^2 - 3$. Then $a_3 = 1, a_2 = \bar{1}, a_1 = \bar{1}$, and $a_0 = 0$.

10 Prime Notation

Here is perhaps the most interesting of the four methods of enumeration. We are all well aware of the uniqueness of prime factorization of positive integers. If we agree to write all the primes in every factorization, making use of the fact that $p^0 = 1$, we get a representation of positive integers. $1 = 2^0, 2 = 2^1, 3 = 3^1 2^0, 4 = 2^2, 5 = 5^1 3^0 2^0$, and $6 = 3^1 2^1$. Now write the list of exponents in the same order as above: $1 = 0_p, 2 = 1_p, 3 = 10_p, 4 = 2_p, 5 = 100_p$, and $6 = 11_p$. In this system, multiplication is especially easy. For example, $12101_p \times 21001_p = 33102_p$. Can you figure how we did this?

11 Problems

1. Each column in the table provides the representation of the numbers from 1 to 12 for a certain system of enumeration. Of course, the entries in column A are decimal representations. Study the pattern, and

determine the method of representation. Then replace the ?'s with the appropriate representations.

A	B	C	D	E	F
1	0	1	1	1	1
2	1	10	10	2	110
3	10	100	11	10	111
4	2	101	20	11	100
5	100	1000	21	12	101
6	11	1001	100	20	11010
7	1000	1010	101	21	11011
8	3	10000	110	22	11000
9	20	10001	111	100	11001
10	101	10010	120	101	11110
11	10000	10100	121	102	11111
12	12	10101	200	110	11100
13	?	?	?	?	?
14	?	?	?	?	?
15	?	?	?	?	?
16	?	?	?	?	?
17	?	?	?	?	?
18	?	?	?	?	?

2. Recall that a base -4 representation of a number is a string of digits $\{0, 1, 2, 3\}$ representing a sum of multiples of powers of -4. For example,

$$\begin{aligned}
321.21_{-4} &= 3(-4)^2 + 2(-4)^1 + 1(-4)^0 + 2(-4)-1 + 1(-4)^{-2} \\
&= 3 \cdot 16 - 2 \cdot 4 + 1 \cdot 1 - 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{16} \\
&= 48 - 8 + 1 - \frac{1}{2} + \frac{1}{16} \\
&= 40 + \frac{9}{16} = 649/16.
\end{aligned}$$

   (a) In the space provided, construct the addition and multiplications tables for the base -4 digits.

+	0	1	2	3
0				
1				
2				
3				

×	0	1	2	3
0				
1				
2				
3				

(b) List, in ascending order, the representations of the integers from 1 to 14.

(c) Note that 12_{-4} represents $1 \cdot (-4)^1 + 2 \cdot (-4)^0 = -2$. List, in descending order, the representations of the first 14 negative integers.

(d) How can one determine whether a number in the system is positive or negative?

(e) Note that the sum of digits table, above, indicates that every carry from addition involves two carry digits. Use this fact to explain why the sum of two 'positives' is 'positive' and the sum of two 'negatives' is 'negative'.

(f) Use the notion that the product of two positive integers may be regarded as successive addition and use your explanation from (5) to argue that the product of two 'positives' is 'positive' and a 'positive' times a 'negative' is 'negative'.

Definition. If two symbols represent the integers that have a sum of zero, then the two integers are called *additive inverses* of each other. Note that from questions (1) and (2) we have that 1_{-4} and 13_{-4} represent additive inverses since $1 + 13 = 0$.

(g) Find a quick method to determine the additive inverse of a given integer. Then use this method to work the subtraction problem $1132003_{-4} - 1202313_{-4}$.

(h) Interpret 123.32_{-4} as we did above for 321.21_{-4}.

(i) Find the base -4 representation of 99

(j) Find the base -4 representation of 17.5

(k) Find the base -4 representation of $1/2, 1/4$ and $1/16$.

(l) Find the base -4 representation of $1/3$

(m) Devise a method to determine if a given integer is a multiple of 5 based on its base -4 representation. Find a digit d that makes $23231123d_{-4}$ a multiple of 5.

(n) Carry out the arithmetic $2312.12_{-4} + 13202.31_{-4}$.

(o) Write the symbol that represents the additive inverse of the number 1202313_{-4}.

(p) Using your answer from the previous question and the definition of subtraction, rewrite and then solve the addition problem defined by $1132003_{-4} - 1202313_{-4}$.

(q) Carry out the arithmetic $112.3_{-4} \times 33.2_{-4}$.

3. Problem 28 of the 2010 MATHCOUNTS, National Sprint Round, states the following. Infinitely many empty boxes (also called cells or squares), each capable of holding four balls are lined up from right to left. At each step we place a ball in the rightmost box that still has room for it and at the same time, we empty all the boxes to the right of it. How many balls are in the boxes after 2010 steps.