One of my favorite manipulatives is wooden cubes. I prefer one inch cubes, but some younger folks might get along well with smaller sizes. I create problems related to pattern recognition, spacial visualization, polynomials, fractions, and probability. Many problems involve pretty combinations of these ideas.

Today's problems are about probability. Build a $2\times2\times2$ cube using eight unit cubes. Then imagine painting all six faces of the large cube. Cut the painted cube apart and randomly select one of the eight unit cubes, which you then roll like a die. What is the probability that a painted face comes up? Most students will quickly reply with the answer $1/2$. Then I can ask, where do we go from here. Some suggest a $3\times3\times3$ problem, but I defer. Let's try the $1\times1\times1$, I urge. They laugh. Okay, I say, what about the $3\times3\times3$ cube? Some students realize that when the $3\times3\times3$ cube is painted, four types of unit cubes are produced: 1 with no painted faces, 6 (called face cubes) with one face painted, 12 edge cubes with two painted faces and 8 corner cubes with three painted faces. This cube classification leads to the probability

$$\frac{1}{27}\cdot\frac{0}{6}+\frac{6}{27}\cdot\frac{1}{6}+\frac{12}{27}\cdot\frac{2}{6}+\frac{8}{27}\cdot\frac{3}{6}=\frac{1}{27}(0+1+4+4)=\frac{1}{3},$$

using the so-called total probability law. Hmm, $1, 1/2, 1/3$ for the cubes with volume 1, 8 and 27. This must not be a coincidence. What's next. Some students will dive right into the $4\times4\times4$ case. Others will begin to ponder a method easier than cube classification. Does anyone see an easier method? YES, students will exclaim. Every one of the $27\cdot6=162$ faces is equally likely to be selected. How many of these 162 faces are painted? Of course, its $6\cdot9=54$ since each of the six faces has area 9. Now

$$\frac{9\cdot6}{27\cdot6}=\frac{1}{3},$$

and we can see pretty quickly that in the general case of $n\times n\times n$ cubes, we get:

$$\frac{6n^2}{6n^3}=\frac{1}{n}$$

as the probability that a randomly selected face is painted.

Where do we go from here? Let's get out of this cubic rut and talk about rectangular boxes. Build an $a \times b \times c$ block of cubes, with $a \leq b \leq c$. We can ask the same question or we can turn it around. Suppose the the block is painted on all six of its faces, and once again a unit cube is randomly selected. Suppose the probability that a painted face comes up is $2/7$.Find all triplets (a, b, c). Some students will see right away that $a = b = c$ is impossible because $2/7$ is not a unit fraction. What should I tell my little fourth grade friend Lucio who tells me that a $3.5 \times 3.5 \times 3.5$ cube will work? He actually built such a cube from cardboard at home.

Giving students time to experiment and think is important at this stage. Some will begin to experience some confusion. Good. Learning to tolerate confusion is an important step in the learning process. We don't want university students saying 'but wait, you haven't taught us how to do that kind of problem'. Here students need to understand the idea that all the faces are equally likely and not focus on the cubes. So there are $6abc$ faces and the number of painted faces is $2(ab + bc + ac)$. Setting this fraction equal to $2/7$, and simplifying results in the equation

$$\frac{1}{c} + \frac{1}{b} + \frac{1}{a} = \frac{6}{7}.$$

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Now you can argue that $a = 2$ since $a = 1$ makes the left side too big and $a = 3$ makes the left dise too small. Subtracing $1/2$ from both sides yields

$$\frac{1}{c} + \frac{1}{b} = \frac{5}{14}.$$

This has the unique integer solution $b = 3, c = 42$.