Let's talk fractions today. Sadly fractions are the 800 pound gorilla for many middle school children. Failure to understand fractions and lack of fluency with their arithmetic often blocks children's entry to algebra and higher level mathematics.

We discuss three problems on fractions. First, find a fraction between 1/2 and 2/3. Of course there are many crrect answers. I always ask my students to explain how they solved the problem. The most popular answer among teachers and middle school students is 7/12, the midpoint of the segment $[1/2, 2/3]$. Its the most popular answer among people who have been taught how to solve problems like this. Among adults over 50years of age, the most popular answer is 5/8, perhaps because they've worked with rules graded in 8ths, or possibly they were stock market investors before the big board went to penny increments for stocks. My favorite two answers are 5/9 and 3/5. I got the answer 5/9 from a precocious six year old. He followed his answer with the explanation that 5/9 is greater than 5/10 which is 1/2 and 5/9 is less than 6/9 which is 2/3. The wonderful answer 3/5 gives the teacher the opportunity to bring up the **water sharing model**. Two girls Ashley and Betty have a one-liter bottle of water to share with each other while three boys Carl, Dick, and Ethan have two one-litter bottles of water to share among themselves. But the five children meet and agree to share all the water equally. How much does each child get? Its important that each student understands that the new fraction 3/5, called the **mediant** of 1/2 and 2/3 lies between 1/2 and 2/3. More generally, using the symbol Δ, we can write

$$\frac{a}{b} \leq \frac{a}{b} \Delta \frac{c}{d} = \frac{a+c}{b+d} \leq \frac{c}{d}$$

when a, b, c and d are positive integers. Note also that the mediant of two fractions depends on their representations and not their values:

$$\frac{1}{2} \Delta \frac{2}{3} \neq \frac{2}{4} \Delta \frac{2}{3}.$$

This oddity is the basis for the interesting Simpson paradox.

The second problem is actually a sequence of problems. First, ask your students to find a fraction $\frac{a}{b}$ with the two properties a) a and b are different digits and b) $\frac{a}{b}$ is as large as possible but less than 1. It won't take long before students name 8/9 as the fraction. 'Can you prove your answer is best', I ask. I want them to recognize that $1 - a/b$ is as small as it can be. The next problem is to find four distinct a, b, c, and d so that

$$\frac{a}{b} + \frac{c}{d} < 1,$$

and the sum is otherwise as large as possible. Someone will realize that we cannot hope to do better than 71/72, so we should try solving

$$\frac{a}{8} + \frac{c}{9} = \frac{71}{72},$$

which sucombs to the Euclidean algorithm or trial and error. Finally we ask for six different digits $a, b, c, d.e$, and f such that $a/b + c/d + e/f$ is less than 1 but as large as possible. What about the 8 digit problem? For this, replace the 1 with a 2.

The third problem comes from The Art of Problem Solving's Beast Academy, book 5 which just came out. Find the next fraction in the sequence

$$\frac{1}{5}, \frac{1}{3}, \frac{3}{7}, \frac{1}{2}, \frac{5}{9}.$$

One of my fabulous fourth graders looked at the sequence of differences $1/3 - 1/5 = 2/15$, $3/7 - 13 = 2/21$, $1/2 - 37 = 2/28$, and $5/9 - 1/2 = 2/36$, and recognized the the denominators are all triangular number. So the next difference should be $5/9 + 2/45 = 27/45 = 3/5$. Yes, now we see that the sequence is just

$$\frac{1}{5}, \frac{2}{6}, \frac{3}{7}, \frac{4}{8}, \frac{5}{9}, \frac{6}{10},$$

with each fraction reduced to lowest terms.