In recent years, mathematics educators have begun to realize that understanding fractions and fractional arithmetic is the gateway to advanced high school mathematics. Yet, US students continue to do poorly when ranked internationally on fraction arithmetic. For example, consider this problem the 1995 TIMSS Trends in International Mathematics and Science Study:

What is $3/4 + 8/3 + 11/8$? The options were
A. 22/15 B. 43/24 C. 91/24 D. 115/24
More than 42 percent of US eighth graders who worked this problem chose option A. This percent was exceeded only by England. Singapore, Japan, and Belgium all had fewer than 10 percent students with answer A.

Consider the following real situation. A man bought a TV set for $1000, only to return a week later, complaining to the manager that he had bought the sets only because of the advertisement that the price was set ‘at a fraction of the Manufacturers’ Suggested Retail Price (MSRP)’. But the man had seen on the internet that the MSRP was only $900. Well, the manager said, ‘ten ninths is a fraction.’ The story does not end there. The customer won a small claims decision because the judge ruled that the manager was a crook. The common understanding of fraction, the judge said, is a number less than 1.

In the words of comedian Red Skelton: Fractions speak louder than words.

What is important here is to note that CCSS requires a model for fraction that enables both conceptual understanding and computational facility. The suggested model is to first understand unit fractions in more of less the same way we understand place value numbers in building an understanding of decimal representation. A place value number (or special number), is simply a digit times a power of 10. These place value numbers are the atoms of the number system, and we learn how to build the other numbers from them and how these numbers enable computation. Now for unit fractions, we can say that every fraction (at the elementary stage we still do not make the distinction between fraction and rational number as we do below) is a sum of unit fractions:

$$\frac{m}{n} = \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}.$$ 

This is similar to the idea that every positive integer is expressible as a sum of place value numbers. Once we learn to perform the arithmetic operations of place value numbers, we are led naturally to the general case. Such happens with fractions also. For a discussion of place value see the paper I co-authored with Roger Howe: http://www.teachersofindia.org/en/article/five-stages-place-value.

**Fraction versus rational number.** What’s the difference? It’s not an easy question. In fact, the difference is somewhat like the difference between a set of
words on one hand and a sentence on the other. A symbol is a fraction if it is
written a certain way, but a symbol that represents a rational number is a rational
number no matter how it is written. Here are some examples. The symbol \( \frac{1}{7} \) is a
fraction that is not a rational number. On the other hand \( \frac{2}{3} \) is both a fraction and
a rational number. Now 0.75 is a rational number that is not a fraction, so we have
examples of each that is not the other. To get a little deeper, a fraction is a string of
symbols that includes a fraction bar, a numerator and a denominator. These items
may be algebraic expressions or literal numbers. Any real number can be written as
a fraction (just divide by 1). But whether a number if rational depends on its value,
not on the way it is written. What we’re saying is that in the case of fractions,
we are dealing with a syntactic issue, and in case of rational numbers, a semantic
issue, to borrow two terms from computer science. For completeness, we say that a
number is rational if it CAN be represented as a quotient of two integers. So 0.75
is rational because we can find a pair of integers, 3 and 4, whose quotient is 0.75.

Here’s another way to think about the difference. Consider the question ‘Are
these numbers getting bigger or smaller?’

\[ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \]

This apparently amusing question can provoke some serious questions about
what we mean by the word ‘number’. Indeed, there are two aspects of numbers that
often get blurred together: the value of a number and the numeral we write for the
number. By ‘number’ we usually mean the value while the word ‘numeral’ refers to
the symbol we use to communicate the number. So the numbers above are getting
smaller while the numerals are getting bigger. This contrast between symbol and
substance also explains the difference between rational number and fraction. A
fraction is a numeral while a rational number is a number.

Rational Numbers. The most common way to study rational numbers is to
study them all at one time. Let’s begin. A rational number is a number which can
be expressed as a ratio of two integers, \( a/b \) where \( b \neq 0 \). Let \( Q \) denote the set of all
rational numbers. That is,

\[ Q = \{ x \mid x = a/b, a, b \in \mathbb{Z}, b \neq 0 \} \]

where \( \mathbb{Z} \) denotes the set of integers. The following exercises will help you understand
the structure of \( Q \).

1. Prove that the set \( Q \) is closed under addition. That is, prove that for any two
rational numbers \( x = a/b \) and \( y = c/d \), \( x + y \) is a rational number.

2. Prove that the set \( Q \) is closed under multiplication. That is, prove that for
any two rational numbers \( x = a/b \) and \( y = c/d \), \( x \cdot y \) is a rational number.
3. Prove that the number midway between two rational numbers is rational.

4. For this essay, we assume the set $\mathbb{R}$ of real numbers is the set of all positive and negative decimal numbers and the number zero. These decimals have three forms, those that terminate, i.e. have only finitely many non-zero digits, like $1.12500\ldots$; those that repeat like $1.3333\ldots = 4/3$, and those that do not repeat. Prove that all rational numbers of one of the first two types, and vice-versa, any number of the first two types is rational.

5. Let $z$ be a positive irrational number. Prove that there is a positive rational $r$ number less than $z$.

6. Prove that the rational numbers $\mathbb{Q}$ is dense in the set of real numbers $\mathbb{R}$. That is, prove that between any two real numbers, there is a rational number.

In the following problems, we need the notion of unit fraction. A unit fraction is a fraction of the form $1/n$ where $n$ is a positive integer. Thus, the unit fractions are $1/1, 1/2, 1/3, \ldots$.

1. **Fractions as Addresses** Divide the unit interval into $n$-ths and also into $m$-ths for selected, not too large, choices of $n$ and $m$, and then find the lengths of all the resulting subintervals. For example, for $n = 2, m = 3$, you get $1/3, 1/6, 1/6/1/3$. For $n = 3, m = 4$, you get $1/4, 1/12, 1/6, 1/6, 1/12, 1/3$. Try this for $n = 3$ and $m = 5$. Can you find a finer subdivision into equal intervals that incorporates all the division points for both denominators? I got this problem from Roger Howe.

2. Here’s a problem from *Train Your Brain*, by George Grätzer. ‘It is difficult to subtract fractions in your head’, said John. ‘That’s right’ said Peter, ‘but you know. there are several tricks that can help you. You often get fractions whose numerators are one less that their denominators, for instance,

$$\frac{3}{4} - \frac{1}{2}$$

It’s easy to figure out the difference between two such fractions.

$$\frac{3}{4} - \frac{1}{2} = \frac{4 - 2}{4 \times 2} = \frac{1}{4}.$$  

Another example is $7/8 - 3/4 = (8 - 4)/(8 \cdot 4)$. ‘Simple, right?’ Can you always use this method?

3. Show that every unit fraction can be expressed as the sum of two different unit fractions.
4. Sums of unit fractions

(a) Notice that $2/7$ is expressible as the sum of two unit fractions: $2/7 = 1/4 + 1/28$. But $3/7$ cannot be so expressed. Show that $3/7$ is not the sum of two unit fractions.

(b) There is a conjecture of Erdős that every fraction $4/n$ where $n \geq 3$ can be written as the sum of three unit fractions with different denominators. Verify the Erdős conjecture for $n = 23, 24, \text{and } 25$.

(c) Can you write 1 as a sum of different unit fractions all with odd denominators?

(d) Can any rational number $r$, $0 < r < 1$ be represented as a sum of unit fractions?

5. In the Space Between

(a) Name a fraction between $1/2$ and $2/3$. Give an argument that your fraction satisfies the condition.

(b) Name a fraction between $11/15$ and $7/10$. How about $6/7$ and $11/13$?

(c) Name the fraction with smallest denominator between $11/15$ and $7/10$. Or $6/7$ and $11/13$?

(d) First draw red marks to divide a long straight board into 7 equal pieces. Then you draw green marks to divide the same board into 13 equal pieces. Finally you decide to cut the board into $7 + 13 = 20$ equal pieces. How many marks are on each piece?

(e) A bicycle team of 7 people brings 6 water bottles, while another team of 13 people brings 11 water bottles. What happens when they share? Some of this material is from Josh Zucker’s notes on fractions taken from a workshop for teachers at American Institute of Mathematics, summer 2009. Some of the material is from the book Algebra by Gelfand and Shen.

6. Dividing Horses

This problem comes from Dude, Can You Count?, by Christian Constanda. An old cowboy dies and his three sons are called to the attorney’s office for the reading of the will.

All I have in this world I leave to my three sons, and all I have is just a few horses. To my oldest son, who has been a great help to me and done a lot of hard work, I bequeath half my horses. To
my second son, who has also been helpful but worked a little less, I bequeath a third of my horses, and to my youngest son, who likes drinking and womanizing and hasn’t helped me one bit, I leave one ninth of my horses. This is my last will and testament.

The sons go back to the corral and count the horses, wanting to divide them according to their pa’s exact wishes. But they run into trouble right away when they see that there are 17 horses in all and that they cannot do a proper division. The oldest son, who is entitled to half—that is $8\frac{1}{2}$ horses—wants to take 9. His brothers immediately protest and say that he cannot take more than that which he is entitled to, even if it means calling the butcher. Just as they are about to have a fight, a stranger rides up and agrees to help. They explain to him the problem. Then the stranger dismounts, lets his horse mingle with the others, and says “Now there are 18 horses in the corral, which is a much better number to split up. Your share is half” he says to the oldest son, “and your share is six”, he says to the second. “Now the third son can have one ninth of 18, which is two, and there is $18 - 9 - 6 - 2 = 1$ left over. The stranger gets on the 18th horse and rides away. How was this kind of division possible.

7. Consider the equation

$$ \frac{1}{a} + \frac{1}{b} = \frac{5}{12}. $$

Find all the ordered pairs $(a, b)$ of real number solutions.

8. Suppose \(\{a, b, c, d\} = \{1, 2, 3, 4\}\).

   (a) What is the smallest possible value of

   $$ a + \frac{1}{b + \frac{1}{c + \frac{1}{d}}}. $$

   (b) What is the largest possible value of

   $$ a + \frac{1}{b + \frac{1}{c + \frac{1}{d}}}. $$

9. **Smallest Sum.** Using each of the four numbers 96, 97, 98, and 99, build two fractions whose sum is as small as possible. As an example, you might try $99/96 + 97/98$ but that is not the smallest sum. This problem is due to Sam Vandervelt. Extend this problem as follows. Suppose $0 < a < b < c < d$ are
all integers. What is the smallest possible sum of two fractions that use each integer as a numerator or denominator? What is the largest such sum? What if we have six integers, $0 < a < b < c < d < e < f$. Now here’s a sequence of easier problems that might help answer the ones above.

(a) How many fractions $a/b$ can be built with $a, b \in \{1, 2, 3, 4\}$, and $a \neq b$?
(b) How many of the fractions in (a) are less than 1?
(c) What is the smallest number of the form $\frac{a}{b} + \frac{c}{d}$, where $\{a, b, c, d\} = \{1, 2, 3, 4\}$?
(d) What is the largest number of the form $\frac{a}{b} + \frac{c}{d}$, where $\{a, b, c, d\} = \{1, 2, 3, 4\}$?

10. **Simpsons** (with thanks to http://www.cut-the-knot.com) Bart and Lisa shoot free throws in two practice sessions to see who gets to start in tonight’s game. Bart makes 5 out of 11 in the first session while Lisa makes 3 out of 7. Who has the better percentage? Is it possible that Bart shoots the better percentage again in the second session, yet overall Lisa has a higher percentage of made free throws? The answer is yes! This phenomenon is called Simpson’s Paradox.

(a) Find a pair of fractions $a/b$ for Bart and $k/l$ for Lisa such that $a/b > k/l$ and yet, Lisa’s percentage overall is better.

The numbers $12/21$ and $11/20$ are called mediants. Specifically, given two fractions $a/b$ and $c/d$, where all of $a, b, c,$ and $d$ are positive integers, the mediant of $a/b$ and $c/d$ is the fraction $(a + c)/(b + d)$.

(b) Why is the mediant of two fractions always between them?

(c) Notice that the mediant of two fractions depends on the way they are represented and not just on their value. Explain Simpson’s Paradox in terms of mediants.

(d) Define the mediant $M$ of two fractions $a/b$ and $c/d$ with the notation $M(a/b, c/d)$. So $M(a/b, c/d) = (a + c)/(b + d)$. This operation is sometimes called ‘student addition’ because many students think this would be a good way to add fractions. Compute the mediants $M(1/3, 8/9)$ and $M(4/9, 2/2)$ and compare each mediant with the midpoint of the two fractions.

Now let’s see what the paradox looks like geometrically on the number line. Here, $B_1$ and $B_2$ represent Bart’s fractions, $L_1, L_2$ Lisa’s fractions,
and $M_B, M_L$ the two mediants.

$\begin{array}{cccc}
L_1 & B_1 & M_B & M_L & L_2 & B_2 \\
\end{array}$

(e) (Bart wins) Name two fractions $B_1 = a/b$ and $B_2 = c/d$ satisfying $0 < a/b < 1/2 < c/d < 1$. Then find two more fractions $L_1 = s/t$ and $L_2 = u/v$ such that

i. $a/b < s/t < 1/2$,

ii. $s/t < c/d < 1$,

iii. $s+t \leq a+c < b+d$.

(f) (Lisa wins) Name two fractions $B_1 = a/b$ and $B_2 = c/d$ satisfying $0 < a/b < 1/2 < c/d < 1$. Then find two more fractions $L_1 = s/t$ and $L_2 = u/v$ such that

i. $s/t < a/b < 1/2$,

ii. $u/v < c/d < 1$,

iii. $s+t \leq a+c < b+d$.

11. Using the notation $M(a/b, c/d)$ we introduced above, write the meaning of each of the statements below and prove them or provide a counter example.

(a) The mediant operation is commutative.

(b) The mediant operation is associative.

(c) Multiplication distributes of ‘mediation’.

12. The number of female employees in a company is more than 60% and less than 65% of the total respectively. Determine the minimum number of employees overall.

13. The fraction of female employees in a company is more than $6/11$ and less than $4/7$ of the total respectively. Determine the minimum number of employees overall.

14. The number of female employees in a company is more than 60% and less than 65% of the total respectively. Determine the minimum number of employees overall.

15. For positive integers $m$ and $n$, the decimal representation for the fraction $m/n$ begins $0.711$ followed by other digits. Find the least possible value for $n$.

16. **Fabulous Fractions**
(a) Use each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once to build some fractions whose sum is 1. For example, \( \frac{3}{7} + \frac{8}{56} + \frac{21}{49} = \frac{3}{7} + \frac{1}{7} + \frac{3}{7} = 1 \). Find all solutions.

(b) Use each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once to fill in the boxes so that the arithmetic is correct.

\[
\begin{array}{c}
\square \\
\square \times \square + \square \times \square + \square = 1.
\end{array}
\]

What is the largest of the three fractions?

(c) Find four different decimal digits \( a, b, c, d \) so that \( \frac{a}{5} + \frac{c}{7} < 1 \) and is as close to 1 as possible. Prove that your answer is the largest such number less than 1.

(d) Next find six different decimal digits \( a, b, c, d, e, f \) so that \( \frac{a}{5} + \frac{c}{7} < 1 \) and the sum is as large as possible.

(e) Find four different decimal digits \( a, b, c, d \) so that \( \frac{a}{5} + \frac{c}{7} < 2 \) but is otherwise as large as possible. Prove that your answer is correct. Then change the 2 to 3 and to 4.

(f) Next find six different decimal digits \( a, b, c, d, e, f \) so that \( \frac{a}{5} + \frac{c}{7} < 2 \) and the sum is as large as possible. Then change the 2 to 3 and to 4.

(g) Finally find four different decimal digits \( a, b, c, d \) so that \( \frac{a}{5} + \frac{c}{7} > 1 \) but is otherwise as small as possible. Prove that your answer is correct. Then change the 1 to 2 and to 3.

17. Use exactly eight digits to form four two digit numbers \( ab, cd, ef, gh \) so that the sum \( \frac{ab}{cd} + \frac{ef}{gh} \) is as small as possible. As usual, interpret \( ab \) as 10\( a \) + \( b \), etc.

18. Next find six different decimal digits \( a, b, c, d, e, f \) so that \( \frac{a}{5} + \frac{c}{7} = \frac{e}{f} \).

19. **Problems with Four Fractions.** These problems can be very tedious, with lots of checking required. They are not recommended for children.

(a) For each \( i = 1, 2, \ldots, 9 \), use all the digits except \( i \) to solve the equation

\[
\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = N
\]

for some integer \( N \). In other words arrange the eight digits so that the sum of the four fractions is a whole number. For example, when \( i = 8 \) we can write

\[
\frac{9}{1} + \frac{5}{2} + \frac{4}{3} + \frac{7}{6} = 14.
\]
(b) What is the smallest integer $k$ such that $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = k$? Which digit is left out?

(c) What is the largest integer $k$ such that $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = k$? Which digit is left out?

(d) For what $i$ do we get the greatest number of integers $N$ for which $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = N$, where $S_i = \{a, b, c, d, e, f, g, h\}$?

(e) Consider the fractional part of the fractions. Each solution of $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = N$ belongs to a class of solutions with the same set of fractional parts. For example, $5/2 + 8/4 + 7/6 + 3/9 = 6$ and $5/1 + 7/6 + 4/8 + 3/9 = 7$ both have fractional parts sets $\{1/2, 1/3, 1/6\}$. How many different fractional parts multisets are there?

(f) Find all solutions to
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = i \]
where each letter represents a different nonzero digit.

(g) Find all solutions to
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 2i \]
where each letter represents a different nonzero digit.

(h) Find all solutions to
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 3i \]
where each letter represents a different nonzero digit.

(i) Find all solutions to
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 4i \]
where each letter represents a different nonzero digit.

(j) Find all solutions to
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} = 5i \]
where each letter represents a different nonzero digit.

(k) Find the maximum integer value of
\[ \frac{a}{b} + \frac{c}{d} + \frac{e}{f} + \frac{g}{h} - i \]
where each letter represents a different nonzero digit.