Information for Teachers

The Julia Robinson Mathematics and Computing Festival prize problems on the next few pages are designed for your students. They are hard problems that your students might find interesting. If they do find them interesting, the problems have accomplished their purpose. Students do not have to work these problems to be successful. In fact, we would like you to pick a prize-winning student even if your students do not solve these problems. We hope you will register your prize winner at the website http://education.uncc.edu/oee/jrmf/, and then urge them to attend. You can ask them which of the prizes they would like, a book prize or a t-shirt prize and you can let us know by means of the online web registration. At that time we hope you will agree to help us that day March 24, 2012. Only female middle school students are invited to take part that day.
1. You’re given the four digits 1, 4, 5, and 6. Your goal is to build as many of the positive integers in the range 1 to 100 as you can. You can use only five operations, +, −, ÷, × and ⋆. The first four are the usual arithmetic operators that you’ve practiced in school. The ⋆ operator is also familiar to you. Its called concatenation, and is defined as follows: \( a \star b = 10a + b \). So, for example, 97 = (5 ⋆ 6) + (4 ⋆ 1). A special prize will go to the student who writes the most numbers in the range 1 to 100 using each of the four digits and using a combination of the five operations. Here’s a start: 1 = (4 + 6) ÷ 5 − 1. Remember that you must use all four digits each time. In case of a tie, the student who uses the ⋆ operator the fewest number of times is the winner.

2. Which would you rather have, 10 pounds of dimes or 100 pounds of pennies?

3. Choose any positive integer as the starting value, and choose a different positive integer as the ending value. Then, perform any of the following moves on the starting number
   
   (a) Add 1.
   (b) Subtract 1.
   (c) Multiply by 10.
   (d) Divide by 10.

   Continue to perform moves until you’ve reached the ending number. For example, you can get from 8 to 71 with the following sequence of moves: Subtract 1: \( 8 - 1 = 7 \). Multiply by 10: \( 7 \times 10 = 70 \). Add 1: \( 70 + 1 = 71 \). Find the fewest moves required to get from 23 to 360; from 11 to 85; from 189 to 12. Notice that we have not said whether we allow non-integers during the process. Try this puzzle first by allowing only integers during the process. Then change the rules to allow any numbers during the process.

4. A bottle and a cork cost $1.10. The bottle cost $1 more than the cork. How much does the cork cost?

5. A bar of soap balances 3/4 of a bar of soap and 3/4 of a pound. How much does the bar of soap weigh?
6. I have fifteen cards numbered 1-15. I put down seven of them face down in a row. Can you use the following information to find out which cards they are? The numbers on the first two cards add to 15. The numbers on the second and third cards add to 20. The numbers on the third and fourth cards add to 23. The numbers on the fourth and fifth cards add to 16. The numbers on the fifth and sixth cards add to 18. The numbers on the sixth and seventh cards add to 21. What are my cards? Can you find any other solutions? How do you know you’ve found all the different solutions?

7. **In the Space Between**

   (a) Name a fraction between 1/2 and 2/3. Give an argument that your fraction satisfies the condition.

   (b) Name a fraction between 11/15 and 7/10. How about 6/7 and 11/13?

   (c) Name the fraction with smallest denominator between 11/15 and 7/10. Or 6/7 and 11/13?

   (d) First draw red marks to divide a long straight board into 7 equal pieces. Then you draw green marks to divide the same board into 13 equal pieces. Finally you decide to cut the board into 7 + 13 = 20 equal pieces. How many marks are on each piece?

   (e) A bicycle team of 7 people brings 6 water bottles, while another team of 13 people brings 11 water bottles. What happens when they share?

8. We have a two-pan balance mechanism. We want to select a set of four positive integer weights. For each set of weights, we want to be able to weigh items which weigh 1, 2, 3, etc. up to some number, the higher, the better. For example, suppose we picked the set of four weights as follows: 2, 2, 5, 11. In this case, we can weigh anything from 1 up to 16. Here’s how. 1 = 5 − 2 − 2; 2 = 2; 3 = 5 − 2; 4 = 2 + 2; 5 = 5; 6 = 11 − 5; 7 = 5 + 2; 8 = 11 + 2 − 5; 9 = 5 + 2 + 2; 10 = 11 + 2 + 2 − 5; 11 = 11; 12 = 11 + 5 − 2 − 2; 13 = 11 + 2; 14 = 11 + 5 − 2; 15 = 11 + 2 + 2; 16 = 11 + 5; but there is no way to make 17.
9. Alison and Charlie are playing a divisibility game with a set of 0 to 9 digit cards.

They take turns choosing a card to the right of the cards that are already there. After two cards have been placed, the two-digit number must be divisible by 2. After three cards have been placed, the three-digit number must be divisible by 3. After four cards have been placed, the four-digit number must be divisible by 4. And so on! They keep taking it in turns until one of them gets stuck.

For example, Alison places the 5. Charlie puts down the 8 to make 58, which is a multiple of 2. Alison puts down the 2 to make 582, which is a multiple of 3. Charlie puts down the 0 to make 5820, which is a multiple of 4. Alison now has to choose from 1, 3, 4, 6, 7, or 9 to make a multiple of 5. Convince yourself that Alison is stuck, and that Charlie has won.

Play the game a few times on your own or with a friend.

Are there any good strategies to help you to win?

After a while, Charlie and Alison decide to work together to make the longest number that they possibly can that satisfies the rules of the game.

They very quickly come up with the five-digit number 12365. Can they make their number any longer using the remaining digits? When will they get stuck? What’s the longest number you can make that satisfies the rules of the game?

Is it possible to use all ten digits to create a ten-digit number? Is there more than one solution? Please send us your explanation of the strategies you use to create long numbers.

10. Suppose each of the five points $A, B, C, D,$ and $E$ are joined by a line segment to each of the points $F, G, H, I, J$ and $K$. What is the maximum number of possible intersections of these segments?
For example, if we build the segments $GE$ and $HB$, we get one point $M$. How many points are there like $M$?

11. Suppose we have a bag with 8 cards, six have a 1 written on them and the other two have a 3 on them.
What is the expected value of the number shown when we randomly draw a single card? Note: by expected value we mean the long term average value of the selection. What is the expected value of a draw if I add one 3 card to the bag? What if I add two 3’s? How many 3’s do I have to add to make the expected value equal to 2? How many 3’s do I have to add to make the expected value equal to 2.5?

12. An 8 × 8 checkerboard has alternating black and white squares. How many distinct rectangles, with sides on the grid lines of the checkerboard contain at least 4 black squares?

13. Every day I take the subway from Startville to Endville. Today I arrived at the Startville station to find that my train was just departing. I caught the next train to Endville, where I left the station there at exactly the same time as if I had caught the first train. How did I manage this? The two trains traveled at the same speed, and I myself did not have to rush to make up the lost time.