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EXPLORING PROBLEM-SOLVING IN A MATHEMATICS TEACHERS’ CIRCLE
Abstract

A growing number of Mathematics Teachers’ Circles throughout the country have been started with the aim of cultivating, among middle school teachers, a “culture of problem-solving”. This is accomplished by engaging the teachers, together with mathematicians and mathematics teacher educators, in solving innovative problems on a regular basis. This article describes a typical session in a Circle, touches on key aspects of a Circle, and discusses the reported impact on the teachers and their teaching.
Exploring problem-solving in a mathematics teachers’ circle

Problem-solving is occupies a central place in the NCTM’s Standards (NCTM, 2000). The problem-solving process usually involves problems for which the students do not know the solution method in advance and where new mathematics is learned by engaging in the process. The Standards recommend that the “students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking” (NCTM, 2000, p. 52). Making problem-solving a central part of teaching may be challenging to the teachers with limited experiences in learning and teaching mathematics this way. The Mathematics Teachers’ Circles were developed with the aim of establishing a “culture of problem-solving” among middle-school mathematics teachers, which could then be carried back into their classrooms (American Institute of Mathematics, n.d.). In this article we will understand how this culture of problem-solving is established by examining a local Mathematics Teachers’ Circle in the Southeast of the country.

A brief history

Mathematics Circles originated in Bulgaria more than a hundred years ago and catered to the mathematical development of middle and high school students (Vandervelde, 2009). Later these Math Circles made their way to Russia where communities sought to identify and encourage students who showed mathematical potential (Formin, Gekin, & Itenberg, 1996). Over time, former members of these Mathematics Circles made their way to the United States and sought to continue the tradition, having benefited from them during their time as students. The success of the Mathematics Circles, along with the enthusiasm of some of their teachers led to the creation of the Mathematics Teachers’ Circles [referred to as Circle hereafter] in northern California. The American Institute of Mathematics (AIM) started the first Circle in 2006 and ever since the AIM
has conducted regular workshops for teams interested in starting a Circle in their local area. Currently there are 22 Circles with new ones being added every year (American Institute of Mathematics, n.d.). This article will describe a typical Circle session in the Southeastern part of the United States, examine key features, and discuss the reported impact of the Circle on the participating teachers.

Vignette of a Teachers’ Circle

The session begins with the five teachers and facilitator having a friendly interaction over coffee and bagels. The usual topics of discussion involve interesting events at school and the classroom. New members are usually introduced to the existing members at this point and camaraderie is fostered among all the teachers. As the teachers start taking their seats, the facilitator outlines the Frogs and Toads problem (Berlekamp, Conway, & Guy, 1982; Bogomolny, n.d.) (see Figure 1) and provides an overview for the teachers.

<table>
<thead>
<tr>
<th>Frogs and Toads</th>
</tr>
</thead>
<tbody>
<tr>
<td>The frogs and toads want to swap places in the picture below. Find the minimum number of moves it would take abiding by the following rules.</td>
</tr>
</tbody>
</table>

The rules:
1. A frog or toad can move into an empty space.
2. A frog or toad can hop over 1 frog or 1 toad to an empty space.
3. A frog or toad can only move forward.

In general, find the minimum number of moves for \( f \) frogs and \( t \) toads to swap places? [Assume that there is one blank space separating the groups initially.]

Figure 1.
The facilitator requests the teachers to get into groups and hands out empty boards to track the position and movement of the frogs and toads; with different colored chips to represent each amphibian. I focus on one group comprising of three teachers - Carol, Rita, and Jason. Initially this group explores possible ways that the frogs and toads could move and they hope to see a pattern. Carol looks at the figure that was provided with 3 frogs and 3 toads [(3,3) will be used to denote the number of frogs and toads respectively] and attempts to work out the number of moves by using 3 red and 3 yellow chips (representing 3 frogs and 3 toads). After a few unsuccessful attempts, Jason suggests that they could try the process with a simpler case of (2,2) or even (1,1) and then work their way up. They work out that 3 and 8 moves are needed for the (1,1) and (2,2) cases respectively, and Carol immediately conjectures that the (3,3) case would involve 13 moves. She bases this on her observation that the difference between 3 and 8 was 5 and assumed the difference would remain constant for a growing number of frogs and toads. However, Jason remains skeptical and thinks that there might be something special about the numbers 3 and 8, but is not sure what it could be.

With the growing number of chips involved, the facilitator suggests that the group could follow the lead of the second group by keeping track of the slides and jumps using S’s and J’s. The group work out the strings of Ss and Js for the (1,1), (2,2) and the (3,3) cases (Figure 2).

<table>
<thead>
<tr>
<th>Case</th>
<th>String</th>
<th>Moves</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>SJS</td>
<td>3</td>
</tr>
<tr>
<td>(2,2)</td>
<td>SJSJJSJSJS</td>
<td>8</td>
</tr>
<tr>
<td>(3,3)</td>
<td>SJSJJSJJSJJSJSJS</td>
<td>15</td>
</tr>
</tbody>
</table>

Figure 2.

They also note that there are 15 moves for the (3,3) case rather than 13 which was the initial conjecture. Further, they observe that the pattern is not alternate, but symmetric on both sides of
a ‘center’. For example in the (1,1) case the center is J (2nd move) and in the (2,2) the center is JJ (4th and 5th moves). Once again, based on the (3,3) case, Carol conjectures that the number of moves are growing by odd numbers, so 3 +5=8, 8+7=15, and the next would be 15+9=24 for the (4,4) case. The group tests this conjecture and is excited to see that it is true. On Jason’s suggestion, the group moves on to other cases where there are a different number of frogs and toads like (1,2) and summarize their data in a table (see Figure 3).

<table>
<thead>
<tr>
<th>F</th>
<th>T</th>
<th>Squares</th>
<th>Moves</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>6</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>8</td>
<td>19</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>9</td>
<td>24</td>
</tr>
</tbody>
</table>

Figure 3.

The members focus on the last column and try to observe a pattern in the number of moves as the number of frogs and toads increase. For example, they observe that the differences are 2,3,3,4,4, and 5 respectively. They use this to conjecture that the (4,5) case would be 29. However, they cannot seem to generalize this for f frogs and t toads. The facilitator, who is moving between the groups, points out that the eventual goal is to find out the number of moves in terms of the number of frogs and toads. In order to move them forward, the facilitator poses the question of how they would determine the number of moves required for 100 frogs and 101 toads? An interesting discussion of recursive and closed form formulas ensues as Jason points out why Carol’s method of looking at the differences would be limited when working with a large number of frogs and toads. Jason then adds a column to their existing table to relate the number of frogs and toads to the number of moves (Figure 4). However, the group still seem to focus on the differences (e.g. \([f+t+4]-[f+t+2]=2\)) between the rows rather than observe that the numbers
1,2,4,6 etc. in the last column could be represented as the product of \( f \) and \( t \) with the general expression being \( f + t + ft \) for the total number of moves with \( f \) frogs and \( t \) toads.

<table>
<thead>
<tr>
<th>F (Frogs)</th>
<th>T (Toads)</th>
<th>Squares</th>
<th>Moves</th>
<th>Pattern</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>( f + t + 1 )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>( f + t + 2 )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>( f + t + 4 )</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>6</td>
<td>11</td>
<td>( f + t + 6 )</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>7</td>
<td>15</td>
<td>( f + t + 9 )</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>8</td>
<td>19</td>
<td>( f + t + 12 )</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>9</td>
<td>24</td>
<td>( f + t + 16 )</td>
</tr>
</tbody>
</table>

Figure 4.

Moving around the room and interacting with the group, the facilitator observes that both groups have reached an impasse and decides that better progress may be made in a discussion involving all the teachers. The facilitator gets the groups to briefly relate their progress and sketches this out on the board. While the group I observe focuses on the difference in the total number of moves as they move down the rows, the other group attempts to find a pattern in the Ss and Js that can be generalized. The facilitator elaborates on the recursive and closed form formulas and reiterates with the entire group that the challenge is to find a closed form solution. At this point, Carol and Jason, who are still absorbed with the problem, observe that the numbers 1, 2, 4, 6, 9, 12 and 16 are just a product of the number of frogs and toads respectively and excitedly share this with the others. The facilitator further points out that \( f + t \) represents the number of slides and \( ft \), the number of jumps, and that there are more jumps than slides as \( f \) and \( t \) grow. The final discussion leaves a lot of the teachers impressed as they see how the various parts of the problem come together in the final solution.

Key aspects of the Mathematics Teachers’ Circle

The vignette shows some key patterns that are essential to the problem-solving experience of the Circle. The first concerns the nature of the problems. All the problems are non-routine and the
solution paths are unknown to the teachers at the outset. This in turn sets up an exploratory stage for the teachers as they evaluate possible solution strategies. In some cases, appropriate tools for the exploration are provided; like the board and colored chips in the above vignette. During the course of the exploration teachers are drawn into making conjectures and justifying them to the group. For example, in the above vignette Carol makes conjectures about the number of moves in the different cases and the entire group engages in verifying these conjectures. It is important that the problem challenges the teachers at the appropriate level so that they can use their prior knowledge to engage in the problem. In some cases a brief introduction of a topic provides the needed support for the teachers. By challenging the teachers at the appropriate level, excitement rather than frustration has been a result of their problem-solving attempts. Over time, this contributes to the belief that problem-solving, and mathematics in general, can be a fun and worthwhile endeavor. Lastly, the teachers have the opportunity to learn new mathematics which is embedded in the tasks as they engage in the problem-solving process. In this vignette, the teachers had an opportunity of coming up with a general expression for the number of moves and in the process also had the opportunity to learn the difference between recursive and closed form formulas.

The second key aspect to the Circle is the establishment of a collegial and collaborative environment that fosters risk taking among the teachers. This begins at the outset as the teachers and the facilitators interact informally over coffee and bagels. These interactions are crucial for the new members as they get to know the existing members of the Circle and feel comfortable to share their thinking in the session, which in turn improves the interactions in the groups. The formation of groups is key in reducing the pressure on individual teachers in coming up with the entire solution individually. More importantly, there is a great deal of learning that takes place
through the interactions with other teachers as they try to understand the ideas put forth by another member or verify a proposed conjecture or try to convince others of their way of thinking. The exploratory component of the problems allows for contributions from all the members as they may assume different roles and there is a low level of risk taking which allows for potential contributions from all the teachers in the group. For example, all teachers can participate in a game, or record numbers, or make a conjecture (even if this turns out to be false). Towards the end as the group arrives at a final solution, there is a feeling that the endeavor was a collaborative one. In the vignette, we observe how Jason’s contributions allows for Carol to change her approach and begin with the (1,1) case instead of the (3,3) case. Progress towards the solution was made in a collaborative way as they interacted to generate and build on the common understanding they had of the problem. On many occasions there are mathematicians and mathematics teacher educators present during the Circle and the teachers get to interact with them in the groups.

Finally, the role of the facilitator is important to the working of the Circle and ensures that there is a continuous push for high level of thinking among the teachers. The facilitator will usually allow the teachers to lead the process and intervenes only when there is an impasse. The facilitators usually emphasize collaboration among the teachers, encourage them to make and test conjectures and provide hints only when absolutely necessary. Most of the time the facilitator will move around and listen to the group discussions with the aim of understanding the approaches being taken and supporting the teachers in this endeavor. Since most of the facilitators are mathematicians, the teachers often get to see a different perspective on ways to think and solve mathematics problems. By internalizing thinking processes that they observe in the sessions, teachers can further develop their own problem-solving abilities. The facilitators
also need to choose the proper problems and ensure that the characteristics stated above are adhered to. It is not unusual for new facilitators to participate in a session prior to leading one. Usually these future facilitators will join a group and work together with the teachers, and at the same time observe the moves made by the facilitators during the session. The existing Circles also maintain resources that can be used by a new facilitator (see Appendix).

The impact on teachers

Teachers attending the Circle have reported, through a questionnaire, focus group, and personal interactions, the benefits for them and their teaching. The biggest benefit reported has been to their mathematical learning. For example a sixth grade teacher says “I wanted to come here so that I would be challenged. Because I am not a mathematician, I am an educator and I think I have a good understanding of what younger children need to carry them to the next level but I am not real comfortable going to a real high level [in mathematics], and so it’s very good for me. It’s good for me to be put in that uncomfortable position, to know how my students are feeling when I put them in that uncomfortable position.” For some of the existing members of the Circle who were already engaging their students in problem-solving activities, attending the Circle provided them the support that they were not getting at school. A sixth grade teacher reported that she felt validated by coming to the Circle meetings and this gave her the belief and courage to carry on the problem-solving focus that she had in her classroom. Attending the Circle to get ideas to inspire the students is also another major plus for the teachers. The Circle provides teachers with a continuous stream of good problems that can be adapted to various classroom scenarios. Further, the online resources also give the teachers opportunities to see problems that are assigned in other Circles. A seventh grade teacher says “My goal in coming here is to find inspiration for myself to help me take math instruction beyond the walls of the classroom and
beyond the limits of the textbook and to help give kids a sense of mathematics as a living discipline that engages their minds no matter where they are or what level they are… it’s a process and an activity that can be fun and inspiring and shut out the whole rest of the world while you are engaged in a puzzle or a problem.” Teachers also have very specific ideas about their teaching. For example a sixth grade teacher shared with the group the following insight on the importance of allowing the students to cope with the uncertainty as they do mathematics. “The whole idea of letting a group struggle with a problem and giving them time to do that is something that is not done in the schools very much because of the time. I think that is so powerful, it’s just a great model for anybody involved in the classroom.” Another teacher pointed out how the interesting problems in his class got the students excited, who in turn shared this excitement with their parents. Experiences like this went a long way in making the parents open to a Reform (NCTM, 2000) oriented curriculum. Getting the students excited about mathematics seemed to be a major motivation for the teachers coming to the Circle.

The problems and the collaborative experience of the Circle allowed the teachers to gain ideas for their own classrooms. One teacher discussed his concern, which was mirrored by other teachers, about getting the students to work constructively in groups in the classroom. He pointed to the game that they played in pairs during a Circle session and thought that this would work in his classroom. “You do that [play a game] you finally start getting kids to open up and to say things and to contribute something then you can expand upon that, like we did, and go into more difficult problems.” The teacher recognizes the collaborative effort needed in getting all students to do challenging problems. “We all contributed to solutions. I could not have solved these problems on my own and yet I was able to be a contributor to solving the problems and we did come up with some solutions that we were working on through that, and if I could do that here
within this group, it gives me ideas then I can see how other kids who may not get solutions on their own could help to contribute to solutions and finally be more active in their learning as far as the classroom is concerned.” One teacher reported on the change in her approach to teaching by even assigning problems to which she did not have the answer and explored these jointly with the students. This teacher also mentioned being confident to focus on the students understanding the material rather than just ‘covering’ the syllabus – “It’s given me a little bit of freedom too because I know that its good math going on…and so it has changed my teaching considerably or at least made me work toward that.”

Discussion

By making problem-solving the central focus of the Circles, the teachers are provided with opportunities to engage in non-routine problems and get a first-hand experience of the challenge and thrill of arriving at a solution. The formation of the collaborative groups, and the role of the facilitator gives the teachers a glimpse of what could be possible in their own classrooms. This is especially important in the current environment of high-stakes testing where there is focus on ‘covering’ the content and preparing the students for tests. The interaction with the mathematicians and mathematics teacher educators provides the teachers with a diverse experience in problem-solving that potentially adds to their professional growth. Further, the interaction among the teachers themselves is a special highlight of the Circles as they get to interact with others who are also trying to implement problem-solving in their classroom. The teachers learn about the challenges faced by their colleagues in other schools and the steps that they have taken to overcome these challenges. Gradually, over time, members develop a body of knowledge that is shared with the newer members through the interactions. With a continually
growing number of Circles throughout the country, we eventually aim that all teachers acquire a “culture of problem-solving” that they pass on to their students.
References

1 American Institute of Mathematics. "Math Teachers' Circle Network."

http://www.mathteacherscircle.org/.


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Appendix of Resources for Teachers’ Circles

Mathematics Teachers Circle Network

http://www.theteacherscircle.org/

Charlotte Teacher’s Circle

http://math.uncc.edu/index.php/organizations/charlotte-teachers-circle.html

Berkeley Math Circle

http://mathcircle.berkeley.edu/

Tucson Math Circle

http://ime.math.arizona.edu/circles/teacher.html

New York City Mathematics Teacher’s Circle

http://nymathcircle.org/teachers

Brazos Valley Math Teachers’ Circle

http://www.math.tamu.edu/outreach/BVMTC/

Utah Teachers’ Math Circle

http://uuteacherscircles.wikispaces.com/

A map containing locations of operating and upcoming student and teachers circles can be found at :http://www.mathteacherscircle.org/membercircles.html

National Association of Mathematics Circles website: www.mathcircles.org