1. On the table there are 25 counters. Two people Al and Betty alternate removing counters. Al goes first. Each player can take from 1 to 4 counters on his turn. The person who picks up the last counter loses. Play this a few times with a partner.

We can denote this game with the notation $N'_4(25)$. The 25 is the size of the initial pile, the subscript 4 denotes the size of largest move, which remains constant through the game, and the prime $'$ means that we are playing the misère version of the game. In this case, that means last player loses. In the normal version of the game, the last player wins.

**Solution:** After playing this game a few times, you no doubt realized that you need to leave just one counter for your opponent, in order to make sure that he takes the last counter. We call the position with 1 counter an Oasis position because its a position you want to move to. The position 0 is called a Poison position because if you more there, you are sure to lose. So what we need is a way of splitting the 26 positions $\{0, 1, 2, 3, 4, \ldots, 25\}$ into two sets $\mathcal{P}$ (Poison) and $\mathcal{O}$ (Oasis). The Oasis positions are the ones we want to move to. Let’s try to work out the properties of these two sets. First, we must have 1 in $\mathcal{O}$ because we can see that we must eventually move there in order to win. Next, when we move to an Oasis, we would not want our opponent to be able to move there. And finally, if our opponent has moved to a position $p \in \mathcal{P}$, we must be able to find a move from there to a position in $\mathcal{O}$. OK, so let’s try setting $\mathcal{O} = \{1, 6, 11, 16, 21\}$ and $\mathcal{P}$ the rest of the positions. So $\mathcal{P} = \{0, 2, 3, 4, 5, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24, 25, 26\}$. Can you prove that from any position in $\mathcal{P}$, there is a move to a position in $\mathcal{O}$.

How would you describe that move? Think about how you can characterize the positions of $\mathcal{O}$? Is it a coincidence that the units digits are 1 and 6? No, in fact, the positions of $\mathcal{O}$ are five counters apart because each move $m$ in the range 1 to 4 has a counterpart move $5 - m$ in the range 4 to 1. So suppose you were able to move to 21, which you could do if you have the first move. If you opponent takes $m$ counters, you can take enough so that the total number taken on the two moves is 5. The way to say that is, take $5 - m$. Can a player ever move from a position in $\mathcal{O}$ to another position in $\mathcal{O}$? No, they are too far apart. This decomposition of the set of positions is what we mean by solving the game. That’s the objective for each game we’ll encounter.

2. Two players take turns breaking up a 6 square by 8 square rectangular chocolate bar. They break the bar only at the divisions between the squares. If the bar breaks into several pieces, they keep breaking one piece at a time until only the squares remain. The first player who cannot make a break is the loser. Who will win?
Problems with Games

Solution: Try this game with your class by starting with a 2 by 3 grid of squares. You’ll see that there are very few positions in the tree. In this six by eight game, every game lasts exactly 47 moves since the final position is 48 single squares, and each move increases the number of pieces by 1. Therefore the first player wins.

3. Consider the game $G_2$ which starts with one pile of 20 counters. The rules allow a player to take 1, 2, or 5 counters on each turn. Denote this game by $N(20; 1, 2, 5)$.

\[
\begin{array}{cccccccccccccccc}
20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
\end{array}
\]

Solution:

\[
\begin{array}{cccccccccccccccc}
20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
P & P & O & P & P & O & P & P & O & P & O & P & O & P & O & P & O & P & O & P & O \\
\end{array}
\]

4. Recall the subtraction game in which two players start with two positive integers $a$ and $b$ written on a board. The first player subtracts one of the numbers on the board from a larger one, and write down the new difference. At each stage, the next player finds a positive difference between two numbers that is not already written on the board and writes it on the board. The first player who cannot find a new positive difference loses. For each of the pairs listed below, write down all the numbers that will eventually appear on the board, and use this information to state whether the game will be won by the first player or the second.

(a) 35 and 42

Solution: The numbers we end up with on the board are \{7, 14, 21, 28, 35, 42\}, just the multiples of 7 up to 42. There are four new numbers placed on the board. That means four moves are made, so the second player wins.

(b) 36 and 42

Solution: Similar to the above, we end up with the multiples of 6 up to 42, so an odd number of moves are made in this game (namely, 5). Therefore the first player wins.

(c) 39 and 42

Solution: Similar to the above, we end up with the multiples of 3 up to 42, so an even number of moves (12) are made in this game. Therefore the second player wins.

(d) 40 and 42
Solution: Similar to the above, we end up with the multiples of 2 up to 42, so an odd number of moves (19) are made in this game. Therefore the first player wins.

5. Next, we play a three-person game. The game is presented in so-called characteristic function form. What this means is that each coalition $C$ has a value $v(C)$ subject to conditions

(a) $v(\emptyset) = 0$

(b) If $A \subset B$, the $v(A) \leq v(B)$.

Consider the game with three players, $A, B, C$ such that $v(\emptyset) = v(A) = v(B) = v(C) = 0$, $v(A, B) = 40$, $v(A, C) = 50$, $v(B, C) = 70$ and $v(A, B, C) = 100$. In fully cooperative games players act efficiently when they form a single coalition, the grand coalition. Here the grand coalition is $\{A, B, C\}$. The focus of the game is to find acceptable distributions of the payoff of the grand coalition. Distributions where a player receives less than it could obtain on its own, without cooperating with anyone else, are unacceptable - a condition known as individual rationality. Imputations are distributions that are efficient and are individually rational. Of course in this game all triplets $(a, b, c)$ of non-negative numbers satisfying $a + b + c = 100$ are imputations. We seek to find all of them that satisfy, in addition, $a + b \geq 40$, $a + c \geq 50$ and $b + c \geq 70$. For example, $(20, 30, 50)$ is one such imputation.
6. There are five rational pirates, A, B, C, D and E. They find 100 gold coins. They must decide how to distribute them.

The pirates have a strict order of seniority: A is superior to B, who is superior to C, who is superior to D, who is superior to E.
The pirate world’s rules of distribution are thus: that the most senior pirate should propose a distribution of coins. The pirates, including the proposer, then vote on whether to accept this distribution. If the proposed allocation is approved by a majority or a tie vote, it happens. If not, the proposer is thrown overboard from the pirate ship and dies, and the next most senior pirate makes a new proposal to begin the system again.

Pirates base their decisions on three factors. First of all, each pirate wants to survive. Secondly, each pirate wants to maximize the number of gold coins he receives. Thirdly, each pirate would prefer to throw another overboard, if all other results would otherwise be equal.

Solution: It might be expected intuitively that Pirate A will have to allocate little if any to himself for fear of being voted off so that there are fewer pirates to share between. However, this is as far from the theoretical result as is possible.

This is apparent if we work backwards: if all except D and E have been thrown overboard, D proposes 100 for himself and 0 for E. He has the casting vote, and so this is the allocation.

If there are three left (C, D and E) C knows that D will offer E 0 in the next round; therefore, C has to offer E 1 coin in this round to make E vote with him, and get his allocation through. Therefore, when only three are left the allocation is C:99, D:0, E:1.

If B, C, D and E remain, B knows this when he makes his decision. To avoid being thrown overboard, he can simply offer 1 to D. Because he has the casting vote, the support only by D is sufficient. Thus he proposes B:99, C:0, D:1, E:0. One might consider proposing B:99, C:0, D:0, E:1, as E knows he won’t get more, if any, if he throws B overboard. But, as each pirate is eager to throw each other overboard, E would prefer to kill B, to get the same amount of gold from C.

Assuming A knows all these things, he can count on C and E’s support for the following allocation, which is the final solution:

A: 98 coins B: 0 coins C: 1 coin D: 0 coins E: 1 coin[1] Also, A:98, B:0, C:0, D:1, E:1 or other variants are not good enough, as D would rather throw A overboard to get the same amount of gold from B.