A Complete Solution to the Magic Hexagram Problem

by Harold Reiter and David Ritchie

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Can the numbers from 1 to 19 be placed one number in each position, in such a way that the sum of the entries on each of the hexagram's nine lines is the same?

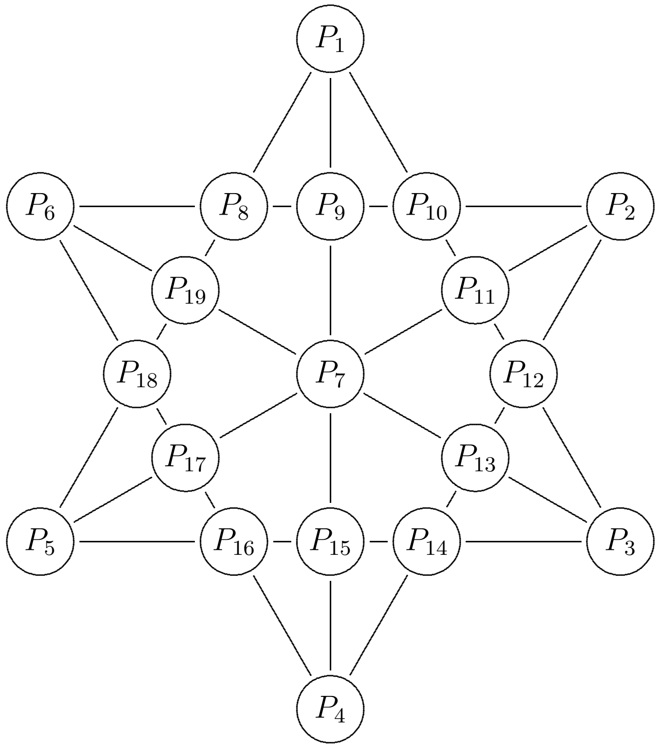


Figure 1.

Martin Gardner showed ([1], [2]) that such an arrangement is possible by producing two "complementary" solutions. (see also [3].) Our objective is to develop an algorithm to find all solutions to this problem. We do this by exploiting the geometric structure of the hexagram and its group of automorphisms. In order to discuss the hexagram more conveniently, let us settle on some notation and terminology. Notice (Figure 1) that the center position and the six outer positions each belongs to three lines, whereas each of the other twelve positions belongs to just two. We shall call the former seven positions *big positions* and the others *small positions.* Thus, the big positions are the ones with labels P_1 to P_7, and the small positions have labels P_8 to P_{19}. The lines

$$L_7 = \{P_1, P_4, P_7, P_9, P_5\}, \qquad L_8 = \{P_2, P_5, P_7, P_{11}, P_{17}\}$$

and

$$L_9 = \{P_3, P_6, P_7, P_{13}, P_{19}\}$$

all contain the center position P_7 and accordingly will be called *center lines*. The *non-center lines* of the hexagram are:

$$\begin{array}{ll} L_1 = \{P_1, P_3, P_{10}, P_{11}, P_{12}\} & L_4 = \{P_2, P_6, P_8, P_9, P_{10}\} \\ L_2 = \{P_3, P_5, P_{14}, P_{15}, P_{16}\} & L_5 = \{P_2, P_4, P_{12}, P_{13}, P_{14}\} \\ L_3 = \{P_1, P_5, P_8, P_{18}, P_{19}\} & L_6 = \{P_4, P_6, P_{16}, P_{17}, P_{18}\}. \end{array}$$

Throughout this exposition $\mathcal{P} = \{P_1, P_2, \ldots, P_{19}\}$ and $\mathcal{L} = \{L_1, L_2, \ldots, L_9\}$. A solution to the hexagram problem is a 1-1 mapping f from $\mathcal{P}$ onto $\mathcal{V} = \{1, 2, \ldots, 19\}$ for which there exists an integer r such that

$$\sum_{P_k \in L_j} f(P_k) = r \text{ for each } L_j \in \mathcal{L}.$$

For a given r, such a function f will be called an r-*solution*.

Every bijection $f : \mathcal{P} \to \mathcal{V}$ satisfies

$$408 = 2\sum_{t=1}^{19} t + \sum_{t=1}^{7} t \le \sum_{j=1}^{9} \left\{ \sum_{p_k \in L_j} f(P_k) \right\} \le 2\sum_{t=1}^{19} t + \sum_{t=13}^{19} t = 492. \qquad (*)$$

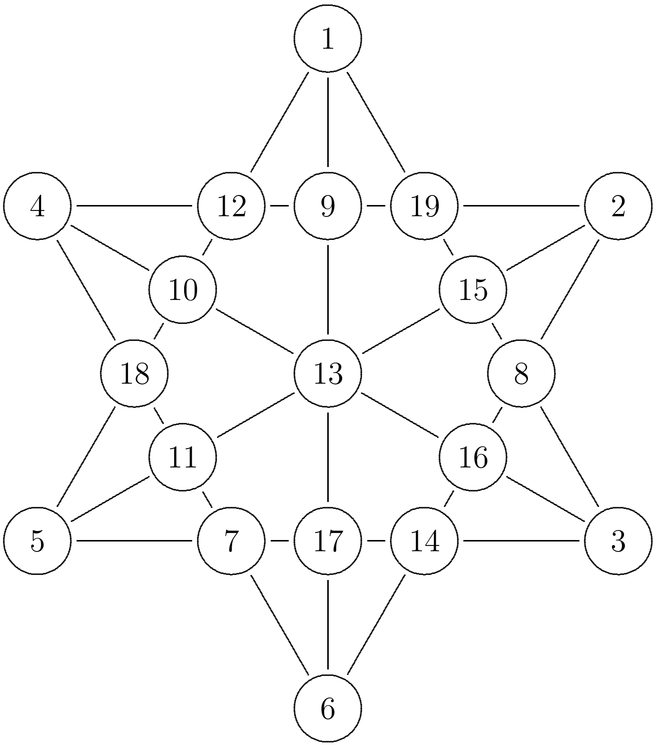


Figure 2a. A 46-solution f

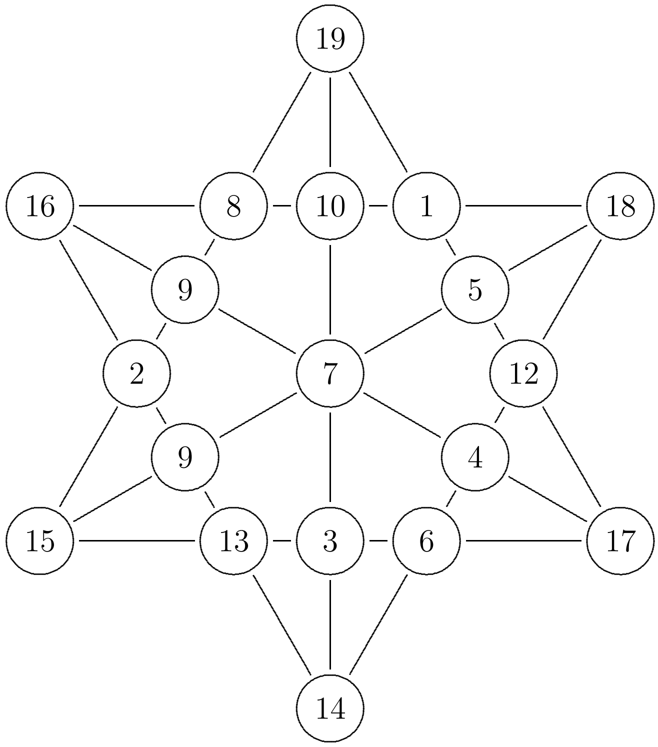


Figure 2b. A 54-solution $\overline{f} = 20 - f$

The lower bound on $\sum_{j=1}^{9}\left\{\sum_{p_k \in L_j} f(P_k)\right\}$ follows by considering any f that assigns to the big positions the numbers 1 through 7, whereas the upper bound follows from assigning 13 through 19 to the big positions. For an r-solution to the hexagram, $(*)$ requires that $408 \leq 9r \leq 492$; that is, $46 \leq r \leq 54$. Note also that $\overline{f} = 20 - f$ is a $(100 - r)$ solution (see Figure 2).

Since $\overline{\overline{f}} = f$, every solution is of the form $\overline{f}$ for some solution f. Therefore, to find all solutions, it is sufficient to find those for which $46 \leq r \leq 50$.

Equivalent r-solutions. Let's consider the group of mappings of P onto P that preserve the hexagram's structure. A one-to-one mapping $h : \mathcal{P} \to \mathcal{P}$ is called an *automorphism* (or *collineation*) of $M = (\mathcal{P}, \mathcal{L})$ if for each $L \in \mathcal{L}$, the set $\{h(P) : P \in L\}$ belongs to $\mathcal{L}$. The set $G(M)$ of automorphisms under composition $\circ$ is a group. As we shall see, this group is isomorphic to the symmetry group G of order 12 of the regular hexagon. (Recall that the elements of G are the six rotations, the three reflections about central lines, and the three reflections about lines joining the midpoints of opposite sides.) To demonstrate this, we begin by confirming that every automorphism h of M takes big positions to big positions, and center lines to center lines. If L^1, L^2, and L^3 are three lines containing a big position Q, then $h(L^1), h(L^2)$, and $h(L^3)$ are three distinct lines containing $h(Q)$, and so $h(Q)$ is a big position. Next, suppose L is a center line containing three big positions–say, Q_1, Q_2, and Q_3. Since $h(Q_1), h(Q_2)$, and $h(Q_3)$ are big positions and since only center lines have three big positions, $h(L)$ must be a center line. From this it follows that h must keep the center position fixed. Of course, the outer big positions are permuted among-themselves. Since each of the positions $P_9, P_{11}, P_{13}, P_{15}, P_{17}$, and P_{19} is situated between P_7 and an outer big position, its image is determined by the image of that outer big position. Similarly, the image of the hexagonal arrangement formed by positions $P_8, P_{10}, P_{12}, P_{14}, P_{16}$, and P_{18} is the hexagonal arrangement itself. Therefore, an automorphism is completely determined once the image of P_1 and the orientation of the outer positions are specified. In other words, each rigid motion of the hexagon with vertices $P_1, P_2, \ldots, P_6$ is the restriction of an automorphism of the hexagram. Conversely, the restriction of any such automorphism to the outer hexagon of positions is a rigid motion of that hexagon.

Suppose that r is fixed and f is an r-solution of the hexagram problem. Then for each automorphism h of M, the composition $f \circ h$ is an r-solution. With this in mind, we say that two r-solutions f and g are equivalent if there exists an automorphism h of M that satisfies $f \circ h = g$. It is an easy exercise to show that this is an equivalence relation on the set of all r-solutions of the hexagram problem. We regard the twelve members $\{f \circ h : h \in G(M)\}$ of each equivalence class as (essentially) the same. Our goal is to determine the number of equivalence classes and to find a representative of each class.

Sorting out the hexagram's diamonds. In this section, we develop the algorithm for solving the hexagram problem. It is convenient to let $x_k = f(P_k)$ for $1 \leq k \leq 19$. Clearly, f is an r-solution if and only if the following integer equations hold:

$$x_1 + x_3 + x_{10} + x_{11} + x_{12} = r \tag{1}$$

$$x_3 + x_5 + x_{14} + x_{15} + x_{16} = r \tag{2}$$

$$x_1 + x_5 + x_8 + x_{18} + x_{19} = r \tag{3}$$

$$x_2 + x_6 + x_8 + x_9 + x_{10} = r \tag{4}$$

$$x_2 + x_4 + x_{12} + x_{13} + x_{14} = r \tag{5}$$

$$x_4 + x_6 + x_{16} + x_{17} + x_{18} = r \tag{6}$$

$$x_1 + x_4 + x_7 + x_9 + x_{15} = r \tag{7}$$

$$x_2 + x_5 + x_7 + x_{11} + x_{17} = r \tag{8}$$

$$x_3 + x_6 + x_7 + x_{13} + x_{19} = r. \tag{9}$$

Note that

$$\sum_{k=1}^{19} x_k = \sum_{i=1}^{19} k = 190. \tag{10}$$

The combination of equations (1) + (3) + (5) + (6) + (7) − (10) gives us

$$2x_1 + 2x_4 + x_{12} + x_{18} = 5r - 190. \tag{11}$$

Similarly, (2) + (3) + (4) + (5) + (8) - (10) yields

$$2x_2 + 2x_5 + x_8 + x_{14} = 5r - 190, \tag{12}$$

and (1) + (2) + (4) + (6) + (9) - (10) yields

$$2x_3 + 2x_6 + x_{10} + x_{16} = 5r - 190. \tag{13}$$

Because of the locations of the positions they involve, equations (11), (12), and (13) are called the *diamond equations*, and each of the solution sets

$$D_1 = \{x_1, x_4, x_{12}, x_{18}\}$$

$$D_2 = \{x_2, x_5, x_8, x_{14}\}$$

$$D_3 = \{x_3, x_6, x_{10}, x_{16}\}$$

will be called a *diamond*. We shall also refer to positions $P_1 - P_6$ as *outside diamond positions*, and $P_8, P_{10}, P_{12}, P_{14}, P_{16}, P_{18}$ as *inside diamond positions*. The advantage of using the diamond equations is that they involve only 12 of the 19 values in $\mathcal{V}$. A brute force search for all 4-element subsets with sum $5r - 190$ ($46 \leq r \leq 50$) is not unreasonable. For each r, it is possible to find the set of all such 4-element sets. (See Table 1.) Each solution f must give rise to three pairwise disjoint diamond sets

$$D_1 = f\{P_1, P_4, P_{12}, P_{18}\}, \qquad D_2 = f\{P_2, P_5, P_8, P_{14}\},$$

and

$$D_3 = f\{P_3, P_6, P_{10}, P_{16}\}.$$

However, not all triplets of diamonds are pairwise disjoint. Let us refer to a triplet of pairwise disjoint diamonds as being a *compatible diamond triplet*. A compatible diamond triplet may still fail to be part of an r-solution because one of its diamonds may contain the number x_7 (note that $f(P_7) = f(P_k)$ for $k \neq 7$ is impossible for a bisection $f : \mathcal{P} \to \mathcal{V}$). To see why x_7 may a number in a diamond, let integers $\{\alpha_1, \beta_1\}$ denote the images of outer diamond positions. Then $\{\alpha_i, \beta_i\} \in \{\{x_1, x_4\}, \{x_2, x_5\}, \{x_3, x_6\}\}$, and $\sum_{i=1}^{3}(\alpha_i + \beta_i) = \sum_{i=1}^{6} x_i$. But by summing equations (1)-(9), we know that $3\sum_{i=1}^{7} x_i + 2\sum_{i=8}^{18} x_i = 9r$. Therefore, rewriting this as $x_7 = 9r - 2\sum_{1=i}^{19} x_i - \sum_{i=1}^{6} x_i$ and using (10), we obtain the following constraint on x_7:

$$x_7 = 9r - 380 - \sum_{i=1}^{3}(\alpha_i + \beta_i). \qquad (**)$$

The preceding remarks can be summarized as follows:

> A bijection $f : \mathcal{P} \to \mathcal{V}$ is an r-solution if and only if its assignments form a compatible diamond triplet $\{D_1, D_2, D_3\}$ such that $x_7 \notin \cup_{i=1}^{3} D_i$ and $\mathcal{V} - (D_1 \cup D_2 \cup D_3)$ can be arranged so that each row of the completed hexagram sums to r.

For $r = 46$, the number of ways to choose 3 diamonds from the set of 269 diamonds is $\binom{269}{3} = 3{,}184{,}242$. A computer search reveals that there are 14,160 compatible diamond triplets. However, 12,081 of these triplets contain x_7 in one of its diamonds. Thus, we need only consider the remaining 2,079 compatible diamond triplets.

At this stage, we must consider the possible geometrical arrangements (configurations) of such a triplet. For definiteness, let's suppose we have such a compatible diamond triplet

$$D_1 = \{\alpha_1, \beta_1, \gamma_1, \delta_1\},$$
$$D_2 = \{\alpha_2, \beta_2, \gamma_2, \delta_2\},$$
$$D_3 = \{\alpha_3, \beta_3, \gamma_3, \delta_3\},$$

where $\{\alpha_i, \beta_i\}$ are to be the images of outside diamond positions and $\{\gamma_i, \delta_i\}$ the images of inside diamond positions. Thus, for each $i = 1, 2, 3$:

$$2\alpha_i + 2\beta_i + 2\gamma_i + 2\delta_i = 5r - 190.$$

We claim that there are 384 mappings

$$f : \{P_1, P_4, P_{12}, P_{18}, P_2, P_5, P_8, P_{14}, P_3, P_6, P_{10}, P_{16}\} \to D_1 \cup D_2 \cup D_3$$

that satisfy the four conditions

$$f(P_1, P_4, P_{12}, P_{18}) = D_i, f(P_2, P_5, P_8, P_{14}) = D_j,$$

$$f(P_3, P_6, P_{10}, P_{16}) = D_k, \text{ and } f(P_1, P_2, P_3, P_4, P_5, P_6) = \{\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3\}$$

for distinct $i, j, k \in \{1, 2, 3\}$.

Table 1. The diamonds for $r = 46$.
Solutions of $2(x + y) + u + v = 40$ for distinct integers $x < y, u, v$ $(1 \leq x, y, u, v \leq 19)$.

$x + y$	x	y	$u + v$	u	v	Number of 4-tuples (x, y, u, v)	Number of 4-tuples with duplicate entries	Number of different diamonds
3	1	2	34	15-16	18-19	2	0	2
4	1	3	32	13-15	17-19	3	0	3
5	1-2	3-4	30	11-14	16-19	8	0	8
6	1-2	4-5	28	9-13	15-19	10	0	10
7	1-3	4-6	26	7-12	14-19	18	0	18
8	1-3	5-7	24	5-11	13-19	21	3	18
9	1-4	5-8	22	3-10	12-19	32	6	26
10	1-4	6-9	20	1-9	10-19	36	8	28
11	1-5	6-10	18	1-8	10-17	40	9	31
12	1-5	7-11	16	1-7	9-15	35	9	26
13	1-6	7-12	14	1-6	8-13	36	11	25
14	1-6	8-13	12	1-5	7-11	30	9	21
15	1-7	8-14	10	1-4	6-9	28	8	20
16	1-7	9-15	8	1-3	5-7	21	6	15
17	1-8	9-16	6	1-2	4-5	16	4	12
18	1-8	10-17	4	1	3	8	2	6
						344	75	269

For a sample calculation, consider the line with $x + y$ value 8. There are 3 ways that two numbers x and y (with $x < y$) can have sum 8: $1 + 7, 2 + 6$ and $3 + 5$. If $x + y = 8$, then $u + v = 40 - 2 \cdot 8 = 24$. There are 7 ways u and v (with $u < v$) can have sum 24. Thus, there are $3 \cdot 7 = 21$ four-tuples. Since three of these — $(1, 7, 7, 17), (2, 6, 6, 18), (3, 5, 5, 19)$ — do not have distinct coordinates, there exist precisely $21 - 3 = 18$ different diamonds.

To verify this, note that α_1 is the image of any of the six positions $P_1, \ldots, P_6$. Then there is no choice for the position whose image is β_1. But then there are two choices of positions for γ_1, (following which there are no alternative choices for δ_1). Having defined the function for one D_i, we proceed to extend to a second set of diamond positions. We have four choices for the position mapped to α_2 and then two choices for γ_2, (the positions associated with β_2 and δ_2 are determined by the choices for α_3 and γ_3). Thus we have a total of $6 \times 2 \times 4 \times 2 \times 2 \times 2 = 384$ functions. Since these functions occur in classes of size 12 (each one is related to 11 others by the equivalence relation discussed earlier), this means there are really just $384/12 = 32$ nonequivalent functions. Thus, these 384 functions are partitioned into 32 equivalence classes of 12 functions per class. To make this process clearer, let us work an example for $r = 46$. First we find three pairwise disjoint 4-element subsets $\{\alpha_i, \beta_i, \gamma_i, \delta_i\}$ of $\mathcal{V}$, each satisfying the diamond equation

$$2\alpha_i + 2\beta_i + 2\gamma_i + 2\delta_i = 40.$$

For this example, a compatible diamond triplet is $\{1, 6, 8, 18\}, \{2, 5, 12, 14\}, \{3, 4, 7, 19\}$. Since the number $x_7 = 9 \cdot 46 - 380 - (1 + 6 + 2 + 5 + 3 + 4) = 13$ computed in $(**)$ does not belong to any of the diamonds in this triplet, we assign the number 13 to P_7 and proceed to the next step. We first assign arbitrarily the values 1 and 6 to P_1 and P_4, respectively. Then, again arbitrarily, let $f(P_{12}) = 8$ and $f(P_8) = 18$. (Notice that an automorphism could be used to rearrange 1 and 6, and 8 and 18, in any of the four possible pairings.) Now we must assign one of the other 4-element sets to the diamond positions $\{P_2, P_5, P_8, P_{14}\}$. Suppose we choose $\{2, 5, 12, 14\}$. Then either $f(P_2) = 2$ or $f(P_2) = 5$, since $\{2, 5\}$ is an $\{\alpha, \beta\}$ pair, and P_2 is an outside position.

Let's set $f(P_2) = 2$. Then $f(P_5) = 5$. Next choose $f(P_8)$ to be either 12 or 14. If we let $f(P_8) = 12$, then we must have $f(P_{14}) = 14$. Finally, we must decide on either 3 or 4 for $f(P_3)$, and 7 or 19 for $f(P10)$. Suppose we choose $f(P_3) = 3$ and $f(P10) = 19$. Having made these choices, it remains to complete the partial hexagram (see Figure 3).

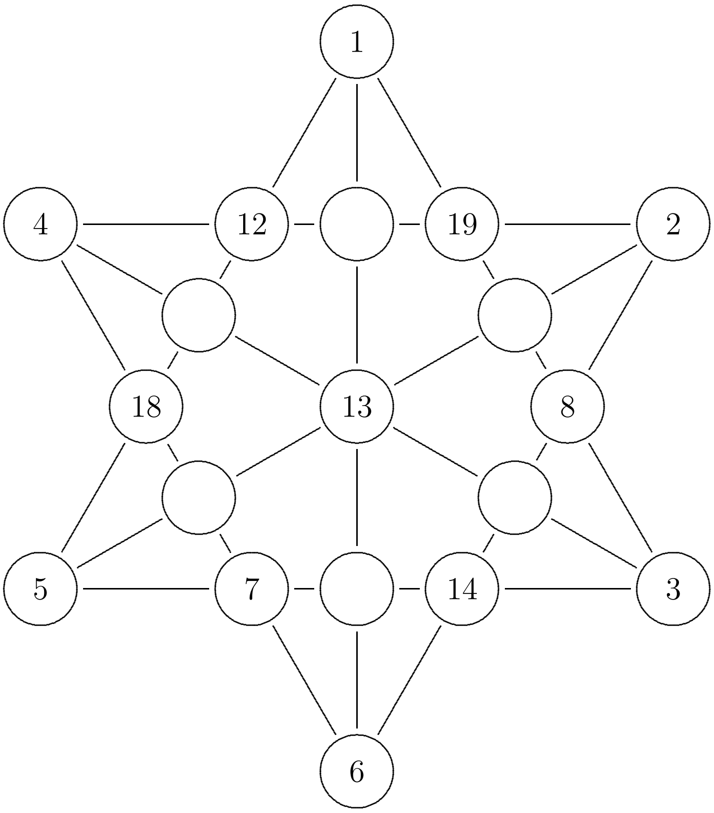


Figure 3. A Partially Completed 46-solution.

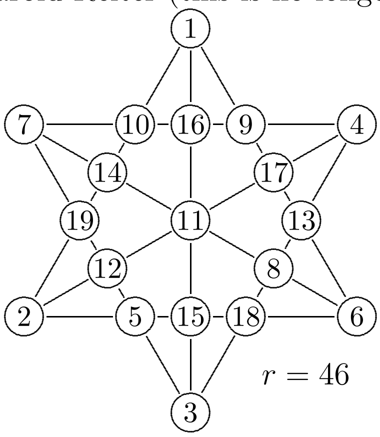
From here it is a simple matter to determine whether the partially defined function f is extendable to a 46-solution. The six unused members $\{9, 10, 11, 15, 16, 17\}$ of the set $\mathcal{V}$ must now fill in the six positions $\{P_9, P_{11}, P_{13}, P_{15}, P_{17}, P_{19}\}$ in such a way that the sum of the entries on each line is $r = 46$. Figure 2a shows that this is indeed possible.

Using our algorithm. As noted earlier, other 46-solutions also exist. When all compatible diamond triplets have been tested and all 46-solutions found, the algorithm calls for the next value of r and the entire search process is repeated for 47-solutions. When all the values of r $(46 < r < 50)$ have been exhausted, the computer program ends. The results are tabulated in Table 2, and some representative solutions follow.

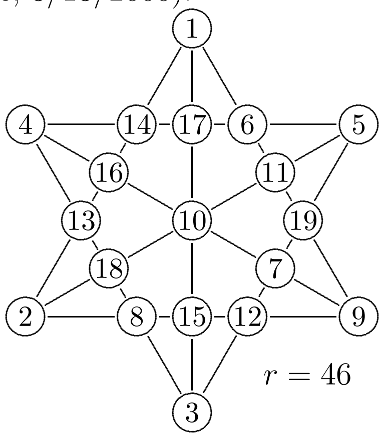
Table 2. Results of the Computer Search*.

r	Number of diamonds	Number of compatible diamond triplets	Number of compatible diamond triplets not containing x_7	Number of r-solutions
46	269	14,160	2,079	91
47	402	131,079	26,831	284
48	438	291,766	72,474	377
49	549	806,008	213,80	888
50	528	821,210	255,588	1100
51	549	806,008	213,800	888
52	438	291,766	72,474	377
53	402	131,079	26,831	284
54	269	14,160	2,079	91
				4380

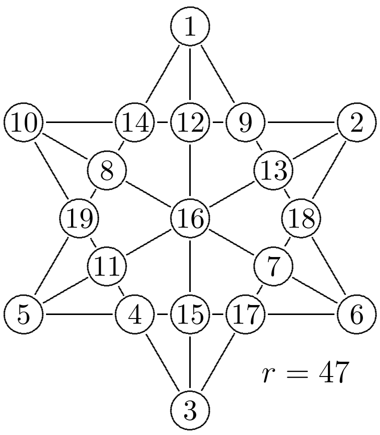
*A copy of the Pascal program used to generate all the solutions is available from Harold Reiter (this is no longer true, 3/13/2000).



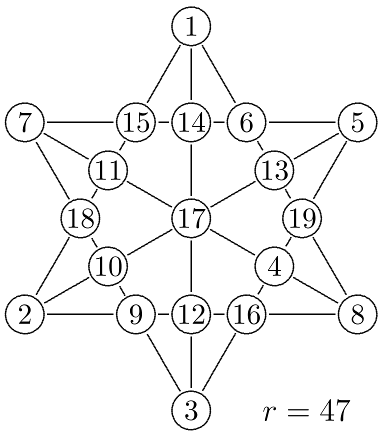
$r = 46$



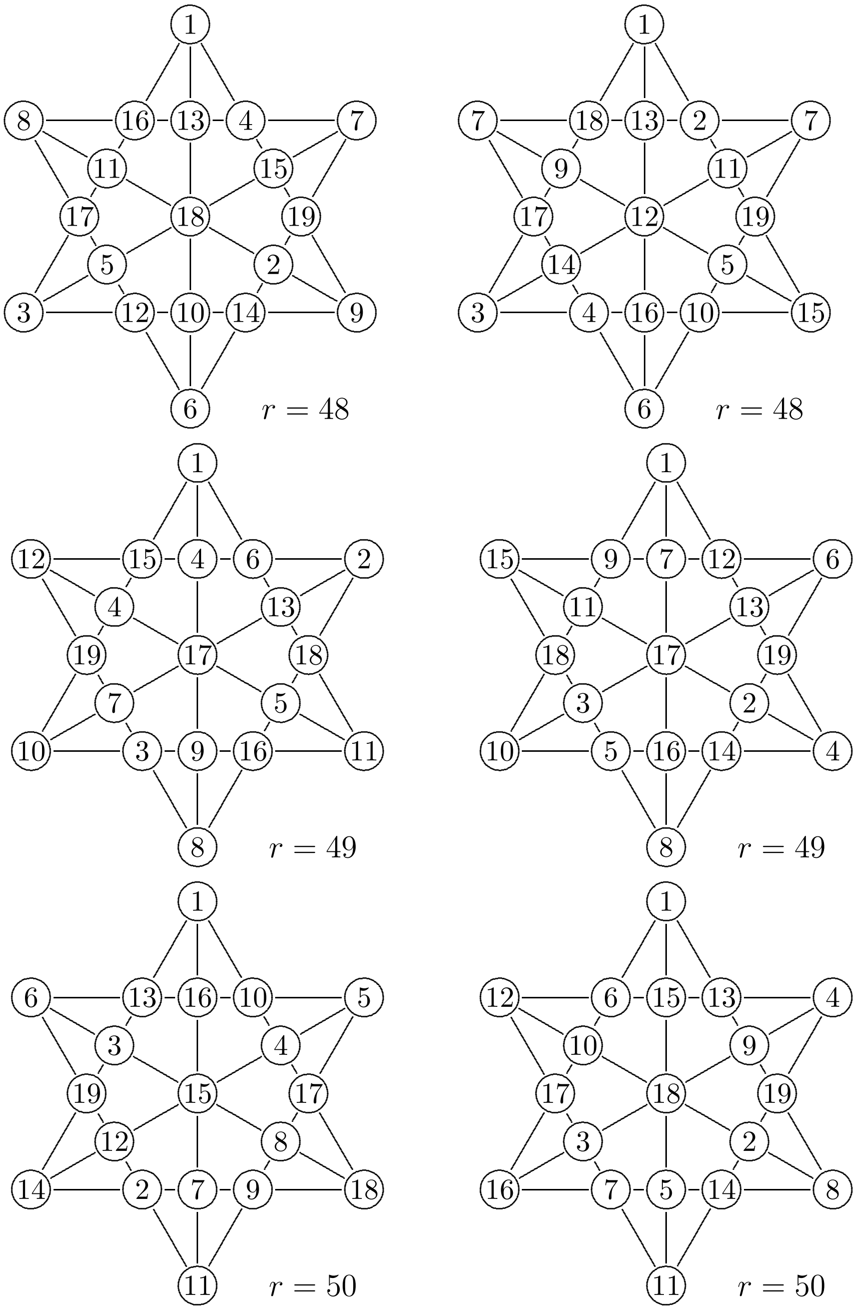
$r = 46$



$r = 47$



$r = 47$



Has this mathematical approach saved much work over the brute force approach of examining all possible configurations? Absolutely! Even if our computer could examine 1,000,000 configurations per second (each of which would require as many 36 binary additions), more than 320 years of nonstop work would be required to examine all 19!/12 possible solution configurations. Our algorithm, on the other hand, requires only 18,264,704 checks. Recalling that each r-solution is the complement of a (100-r)-solution, the algorithm need only consider r = 46 through r = 50 in checking each of the 32 configurations for each of the compatible diamond triplets not containing x_7. That is, it suffices to examine only

$$32(2079 + 26,831 + 72,474 + 213,800 + 255,588) = 18,264,704$$

cases, and this uses less than 5 minutes of CPU time!

Additional considerations. It is easy to see that any nineteen consecutive integers $x_1, x_1+1, x_1+2, \ldots, x_1+18$ can be distributed in the nineteen positions $P_1, \ldots, P_{19}$, one number per position, so that the sum along each of the nine lines is the'same. Suppose $\mathcal{V} = (1, 2, \ldots, u-1, u+1, \ldots 20)$, where $2 \leq u \leq 19$. Can the members of $\mathcal{V}$ be arranged in the desired way? Suppose we expand our notion of solution to include all functions $f : P \to G$, where G is a group, relaxing the requirement that f be a bisection and simply requiring that for some $c \in G$,

$$\sum_{P_k \in L_j} f(P_k) = c$$

for each $L_j \in \mathcal{L}$. If we let G be the group of integers modulo 2, the set of solutions is also a group, under addition of functions. It would be interesting to see the order of this group and to investigate its structure.

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