Grundy Value Classification

In this lecture we continue to study counter pickup games. In what follows, we discuss a finer classification of positions. In this classification, called the Grundy value classification, safe positions are those with a label 0 and unsafe positions all have a positive integer classification. Rather than classify positions of a game as safe or unsafe, we assign a numerical value to each position.

Grundy Values of Positions. It is possible, no matter what combinatorial game we are playing, to assign to each position a number, called the Grundy-value or G-value of the position, in such a way that the positions with G-value 0 are precisely those of \( S \), and all those of \( U \) have positive value. To carry this out, we need a function called \( \text{mex} \). The \( m \) stands for \textit{minimum} and the \( ex \) stands for \textit{excludant}. Here’s the definition. If \( T \) is a finite set of nonnegative integers, \( \text{mex}(T) \) denotes the smallest nonnegative integer \( \text{NOT} \) in \( T \). For example,

\[
\begin{align*}
\text{mex } \{1, 2, 3\} & = 0 \\
\text{mex } \{0, 1, 4, 7\} & = 2 \\
\text{mex } \{0, 1, 2, 3\} & = 4 \\
\text{mex } \{5\} & = 0.
\end{align*}
\]

Let’s go back to the digraph representing \( N(10; 1, 2) \), assign position 0 the G-value 0 and continue to work upwards, assigning to each position the mex of all the values accessible from that position. In symbols, if \( p \) is a position and \( S(p) = \{q : \text{there is a move from } p \text{ to } q\} \) is the set of all successors of \( p \), the Grundy-value of the position \( p \), denoted \( G(p) \), is given by

\[
G(p) = \text{mex } \{G(q) : q \in S(p)\}.
\]

Notice that \( G(0) = G(3) = G(6) = G(9) = 0 \). Compute the G-values of the positions in the digraph game of Fig. 4.

At this stage we consider the relationship between the \( S, U \) classification and the Grundy-value classification. We can prove that the Grundy-value represents a finer classification in the following sense: Each position with G-value 0 is an \( S \) position and each positive G-value position belongs to the set \( U \) of unsafe positions. It is a proof by induction. To be rigorous, we introduce some notation. Let \( Z \) and \( P \) represent the positions with 0 and positive G-values, respectively. And as before, let \( S \) and \( U \) represent the safe and unsafe positions in the same directed graph game \( G \). Our claim is that \( S = Z \) and \( U = P \). First, note that all the terminal positions are both in \( S \) and in \( Z \). Now take a position \( p \) all of whose successors have been labeled
in both classifications, and assume that for all these positions, the claim holds. If 
\( p \) belongs to \( S \), then it does so because there is no move to an \( S \) position, which 
means that \( G(p) = \text{mex} \left( \{ G(q) : q \in S(p) \} \right) = \text{mex} \left( T \right) = 0 \), because \( 0 \notin T \). On 
the other hand, if \( p \in U \), then there is a move from \( p \) to a member of \( S \), that is 
to a position with \( G \)-value 0. Hence \( G(p) > 0 \). This proves that the sets \( Z \) and \( S \) 
coincide and that the sets \( P \) and \( U \) coincide.

Next, we learn to play the game \( N(10,11) \). This game is played as follows: When 
it’s your turn, select a nonempty pile and take any number of counters from it. Play 
this game with your partner several times and then try to imagine what the directed 
graph version looks like. This game is an example of Bouton’s Nim, which is played 
with any number of piles of any sizes in the same way. When it’s your turn, select 
a nonempty pile and take any number of counters from it. The winner is, as usual, 
the last player to make a move. The directed graph for the game \( N(10,11) \) has 
\((10 + 1)(11 + 1) = 132\) vertices and 1265 \((= \frac{11}{2} \cdot 10 \cdot 11 + \frac{12}{2} \cdot 11 \cdot 10)\) edges, a rather 
large digraph. To simplify the labeling process, we can draw an \( 11 \times 12 \) grid of 
squares. Imagine the grid located in the plane so that the bottom left corner square 
has \((0,0)\) at its center and the upper left corner square has \((0,11)\) at its center as 
shown below.
We can play the game on the grid itself instead of using two piles of counters. We start with a marker at the square (10, 11). To make a move, slide the marker horizontally to the left or downward. A move to the left corresponds to removing counters from the 10-counter pile, while downward slides correspond to removing counters from the 11-counter pile. Play the game $N(10, 11)$ with your partner. Try to anticipate what comes next. First, we’ll label on the grid provided the positions with the symbols $S$ and $U$, according to whether they are safe or unsafe to move to. One winning strategy for the starting player is clear. You can move from $(10, 11)$ to $(10, 10)$ and continue to restore the symmetry each time your opponent disturbs it (which he must do on each of his plays). Next, use the other copy of the grid provided to establish the Grundy value of each position. To play Bouton’s Nim perfectly, we must understand how to calculate the G-values of these positions quickly. Next we’ll view the table of G-values as the results of an operation $\oplus$ on the nonnegative integers. Turn the table 90 degrees so that the upper left corner is the entry $(0, 0)$. 

$$
\begin{array}{cccc}
(0,11) & & & (10,11) \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
(0,0) & & & (10,0)
\end{array}
$$
There is no need to limit ourselves to a $10 \times 11$ grid. Let’s go for $15 \times 15$ for now, but with the understanding that the operation is defined on the entire set of natural numbers \{0, 1, 2, 3, \ldots\}. Then the grid looks like:

\[
\begin{array}{cccccccccccccccc}
\oplus & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
1 & 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 & 9 & 8 & 11 & 10 & 13 & 12 & 15 & 14 \\
2 & 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 & 10 & 11 & 8 & 9 & 14 & 15 & 12 & 13 \\
3 & 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 & 11 & 10 & 9 & 8 & 15 & 14 & 13 & 12 \\
4 & 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 & 12 & 13 & 14 & 15 & 8 & 9 & 10 & 11 \\
5 & 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 & 13 & 12 & 15 & 14 & 9 & 8 & 11 & 10 \\
6 & 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 & 14 & 15 & 12 & 13 & 10 & 11 & 8 & 9 \\
7 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 \\
8 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
9 & 9 & 8 & 11 & 10 & 13 & 12 & 15 & 14 & 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\
10 & 10 & 11 & 8 & 9 & 14 & 15 & 12 & 13 & 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\
11 & 11 & 10 & 9 & 8 & 15 & 14 & 13 & 12 & 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\
12 & 12 & 13 & 14 & 15 & 8 & 9 & 10 & 11 & 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\
13 & 13 & 12 & 15 & 14 & 9 & 8 & 11 & 10 & 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\
14 & 14 & 15 & 12 & 13 & 10 & 11 & 8 & 9 & 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\
15 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
\end{array}
\]

There is another way to think of $N(10,11)$ which will prove useful later. We can treat it as a pair of single pile games which we agree to play simultaneously. Any finite collection of games can be put together in this way. Such games are called composite games. Here’s how composite games are played. Suppose we have games $G_1$ and $G_2$. Their composite is denoted $G_1 \oplus G_2$. To play $G_1 \oplus G_2$, the first player chooses one of the two component games and makes a move in it. Then the other player chooses one of the games and makes a move in it. This continues until one of the games reaches a terminal position, whereupon both players move in the nonterminated game until it terminates. As usual, the last player making a move wins. Notice that the game $N(10) \oplus N(11)$ is played in just the same way as $N(10,11)$: at each turn a nonempty pile is picked and take some counters are taken from it. Let’s play the composite game $N(12;1,2,5) \oplus N(11;1,2,3)$ to get an idea of how composite games work. To do this, take a pile of 12 counters and another of 11 counters. You’re allowed to remove 1, 2 or 5 from the former pile and 1, 2 or 3 from the later.

At this stage some astute observers have noticed that we have a symbol $\oplus$ which we are using for two purposes, as a binary operation on games and as a binary operation on nonnegative numbers. We’ll have to be careful not to let that confuse us. We will usually be able to tell from context which of the two is meant.
Let’s investigate some of the properties of $\oplus$ on the set $Z^+ = \{0, 1, 2, 3, \ldots\}$. We can see that $(Z^+, \oplus)$ has an identity element $0$. It’s also not hard to see that the operation is associative and commutative. Notice that the 0’s down the diagonal tell us that each element is its own inverse. Hence the pair $(Z^+, \oplus)$ is what is called an Abelian group. There’s another important property as well. Each of the subsets $S_i = \{0, 1, 2, \ldots, 2^i - 1\}$ is a subgroup of $(Z^+, \oplus)$. What this means is that each of these subsets $S_i$ is closed under the $\oplus$ operation. But how do we compute in $(Z^+, \oplus)$? The answer is this:

Suppose we want to compute $u \oplus v$. Write $u$ and $v$ in binary and ‘add’ the two binary numbers using the following digit table:

<table>
<thead>
<tr>
<th>$\oplus$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Let’s translate this into a practical strategy for winning $N(1, 3, 5, 7)$ or any game of Bouton’s Nim. For any position, we construct the binary configuration. For convenience, we’ll go with $N(1, 3, 5, 7)$.

\[
\begin{align*}
1 &= 1 \\
3 &= 1 1 \\
5 &= 1 0 1 \\
7 &= 1 1 1 \\
0 0 0 &= 0
\end{align*}
\]

In other words, apply the addition table to each column individually. If the result in each column is 0, call the configuration balanced, otherwise unbalanced. It turns out that the balanced positions are exactly the $S$ positions. To prove this we have to see

1. that all terminal positions are balanced,
2. that for each unbalanced position $p$, there is a move to a balanced position $q$, and
3. that from any balanced position $q$, every move results in an unbalanced position.

The first item is clear. There is only one terminal position $O = (0, 0, 0, \ldots 0)$, one 0 for each pile, and its binary configuration is certainly balanced.
Next suppose we have an unbalanced position, say \((u_1, u_2, \ldots, u_k)\). This will be much clearer if we deal with actual values. To be definite, take the position \((5, 10, 15, 20, 25)\) whose binary configuration is given by

\[
\begin{align*}
5 &= 1 \ 0 \ 1 \\
10 &= 1 \ 0 \ 1 \ 0 \\
15 &= 1 \ 1 \ 1 \ 1 \\
20 &= 1 \ 0 \ 1 \ 0 \ 0 \\
25 &= 1 \ 1 \ 0 \ 0 \ 1 \\
0 &= 0 \ 1 \ 1 \ 0 \ 1
\end{align*}
\]

That is, the second, third and fifth columns have an odd number of 1’s and the other columns have an even number of 1’s. To move from \((5, 10, 15, 20, 25)\) to a balanced position requires finding a way to balance these three unbalanced columns without disturbing the balanced nature of the first and fourth columns. To change the second column to one with an even number of 1’s means that we have to remove some counters from one of the three piles which contributes a 1 to that column (why?). That is, we must remove counters from either the 10 pile, the 15 pile or the 25 pile. If we select the 25 pile, we must remove just enough counters to change the 0 that pile contributes in the third column to a 1, leave the 0 contribution in the fourth column, and change the 1 in the fifth column to a 0. We want the pile to contribute \(1\ 0\ 1\ 0\ 0\) which is the binary representation of 20. Thus the move \((5, 10, 15, 20, 25) \rightarrow (5, 10, 15, 20, 20)\) takes the game from a position in \(U\) to a position in \(S\). Finally, we need to be sure that any move from a balanced position results in an unbalanced position. This is easy to see. Each move reduces the size of some pile. The binary representation of that pile changes when the pile is reduced. In particular, some 1 gets changed to a 0. The column in which that 1 appears initially is changed to 0 by the move, and this unbalances that column. We’re done. To win at Bouton’s Nim, just analyze your initial position. If its balanced, you’ll probably lose. If its unbalanced, find a balanced position to move to, and after that continue to move to balanced positions.

Now for the really good news. You can win any composite combinatorial game in the same way. Here’s an example that shows how. Suppose we’re playing the composite game \(N(20; 1, 3, 5) \oplus N(20; 1, 2, 5) \oplus N(20; 1, 2, 6) \oplus W(12, 11) \oplus N(3, 5, 7, 9)\) where \(W(12, 11)\) is Wythoff’s game with piles of sizes 12 and 11. Wythoff’s game is played just like two pile nim except that players can also remove the same number of counters from both piles at a turn. See the homework problem on Wythoff’s game. Of course, the game \(N(3, 5, 7, 9)\) is itself a composite of the four one pile nim games \(N(3), N(5), N(7)\) and \(N(9)\). That is, \(N(3, 5, 7, 9) = N(3) \oplus N(5) \oplus N(7) \oplus N(9)\)
N(9). This composite game is played as follows: at each turn a player selects one of the five component games and make a legal move in that game. For example, denoting the initial position by (20, 20, 20, (12, 11), 3, 5, 7, 9), the first player could move to (20, 20, 20, (9, 8), 3, 5, 7, 9), since that corresponds to taking one counter from each of the two Wythoff’s piles. We know from earlier about the Grundy values of the component games, and we are therefore able construct the binary configuration as follows:

<table>
<thead>
<tr>
<th>Component Game</th>
<th>Grundy value</th>
<th>Binary representation of Grundy value</th>
</tr>
</thead>
<tbody>
<tr>
<td>N(20; 1, 3, 5)</td>
<td>0</td>
<td>0 0 0 0 0 0</td>
</tr>
<tr>
<td>N(20; 1, 2, 5)</td>
<td>2</td>
<td>0 0 0 1 0 0</td>
</tr>
<tr>
<td>N(20; 1, 2, 6)</td>
<td>3</td>
<td>0 0 0 1 1 1</td>
</tr>
<tr>
<td>N(3)</td>
<td>3</td>
<td>0 0 0 1 1 1</td>
</tr>
<tr>
<td>N(5)</td>
<td>5</td>
<td>0 0 1 0 1 1</td>
</tr>
<tr>
<td>N(7)</td>
<td>7</td>
<td>0 0 1 1 1 1</td>
</tr>
<tr>
<td>N(9)</td>
<td>9</td>
<td>0 1 0 0 1 1</td>
</tr>
<tr>
<td>W(9, 8)</td>
<td>15</td>
<td>0 1 1 1 1 1</td>
</tr>
<tr>
<td>Composite</td>
<td>?</td>
<td>0 0 1 1 1 0</td>
</tr>
</tbody>
</table>

We may conclude that the Grundy value of the composite game is the $\oplus$ sum (or sometimes called the nim sum) of the component games, which is in this case $110_2 = 6$. To restore the balance to the configuration, we need to select a game which contributes a 1 to the third column. Any of the games $N(5), N(7), \text{or } W(9, 8)$ will work. Can you find the winning move from $W(9, 8)$?

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