Asymptote Theorem

The Asymptote Theorem for Rational Functions

Let \( p(x) \) and \( q(x) \) be polynomial functions, and define \( r(x) \) by \( r(x) = \frac{p(x)}{q(x)} \). Let

\[
p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0
\]

and

\[
q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0.
\]

Also, suppose that \( p(x) \) and \( q(x) \) have no common zeros. Then

(a) Horizontal Asymptote

i. If \( m < n \), there is no horizontal asymptote.

ii. If \( m = n \), then \( y = \frac{a_n}{b_m} \) is the horizontal asymptote.

iii. If \( m > n \), then \( y = 0 \) is the horizontal asymptote.

(b) Vertical Asymptotes

i. Every zero of the denominator \( q(x) \) determines a vertical asymptote. If \( r_1, r_2, \ldots, r_k \) are zeros of \( q(x) \), then the lines \( x = r_1, x = r_2, \ldots, x = r_k \) are all vertical asymptotes of \( r(x) \).