May 4, 2001

The first five problems count 7 points each (total 35 points) and rest count as marked. There are 195 points available. Good luck.

1. Consider the function \( f \) defined by:
\[
f(x) = \begin{cases} 
2x^2 - 3 & \text{if } x < 0 \\
5x - 3 & \text{if } x \geq 0
\end{cases}
\]
Find the slope of the line which goes through the points \((-2, f(-2))\) and \((3, f(3))\).

(A) \( \frac{7}{5} \)
(B) \( 2 \)
(C) \( \frac{17}{5} \)
(D) \( 5 \)
(E) \( 7 \)

Solution: The two points on the graph are \((-2, 5)\) and \((3, 12)\) and the slope of the line joining them is \( m = \frac{7}{5} \).

2. The distance between the point \((6.5, 8.5)\) and the midpoint of the segment joining the points \((2, 3)\) and \((5, 6)\) is

(A) \( \sqrt{22} \)
(B) \( \sqrt{23} \)
(C) \( 5 \)
(D) \( \sqrt{26} \)
(E) \( 6 \)

Solution: The midpoint of the segment is \((3.5, 4.5)\), so the distance is \( d = \sqrt{3.5^2 + 4.5^2} = \sqrt{25} = 5 \).

3. Let \( f(x) = 2x + 3 \) and \( g(x) = 3x - 9 \). Which of the following does not belong to the domain of \( f/g \)?

(A) 1  
(B) 3  
(C) 6  
(D) 9  
(E) The domain of \( f/g \) is the set of all real numbers.

Solution: Only a number for which \( g \) is zero fails to be in the domain. Solving \( 3x - 9 = 0 \) yields \( x = 3 \).

4. The line tangent to the graph of a function \( f \) at the point \((2, 5)\) on the graph also goes through the point \((0, 7)\). What is \( f'(2) \)?

(A) \(-2\)  
(B) \(-1\)  
(C) 0  
(D) 1  
(E) 2

Solution: The slope of the line through \((2, 5)\) and \((0, 7)\) is \(-1\).

5. What is the slope of the tangent line to the graph of \( f(x) = x^{-2} \) at the point \((2,1/4)\)?

(A) \(-1/4\)  
(B) \(-1/8\)  
(C) \(-1/16\)  
(D) \(-1/256\)  
(E) \(-1/512\)

Solution: The derivative is \( f'(x) = -2x^{-3} \) whose value of at \( x = 2 \) is \( f'(2) = -1/4 \).
6. (15 points) Let \( f(x) = 1/(3x) \).

(a) Construct \( \frac{f(2+h)-f(2)}{h} \).

**Solution:**
\[
\frac{f(2+h)-f(2)}{h} = \frac{\frac{1}{3(2+h)} - \frac{1}{3(2)}}{h} = -\frac{1}{(2+h)6}.
\]

(b) Simplify and take the limit of the expression in (a) as \( h \) approaches 0 to find \( f'(2) \).

**Solution:**
\[
\lim_{h \to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h \to 0} -\frac{1}{(2+h)6} = -\frac{1}{12}.
\]

(c) Use the information found in (b) to find an equation for the line tangent to the graph of \( f \) at the point \( (2, 1/6) \).

**Solution:**
\[
y - \frac{1}{6} = -\frac{1}{12}(x-2).
\]

7. (10 points) Find the rate of change of \( f(t) = e^{2t} \cdot \ln(t) \) when \( t = 1 \).

**Solution:** Use the product rule to get \( f'(t) = 2e^{2t} \cdot \ln(t) + (1/t) \cdot e^{2t} \) whose value at \( t = 1 \) is \( f'(1) = 2e^2 \cdot \ln(1) + (1/1) \cdot e^2 = e^2 \).

8. (20 points) Suppose the functions \( f \) and \( g \) are differentiable and their values at certain points are given in the table. The next four problems refer to these functions \( f \) and \( g \). Notice that, for example, the entry 1 in the first row and third column means that \( f'(0) = 1 \). Note also that, for example, if \( K(x) = f(x) - g(x) \), then \( K'(x) = f'(x) - g'(x) \) and \( K'(4) = f'(4) - g'(4) = 5 - 10 = -5 \). Answer each of the questions below about functions that can be build using \( f \) and \( g \).

\[
\begin{array}{cccc}
 x & f(x) & f'(x) & x & g(x) & g'(x) \\
 0 & 2 & 1 & 0 & 5 & 5 \\
 1 & 2 & 3 & 1 & 7 & 3 \\
 2 & 5 & 4 & 2 & 4 & 6 \\
 3 & 1 & 2 & 3 & 2 & 6 \\
 4 & 3 & 5 & 4 & 6 & 10 \\
 5 & 6 & 4 & 5 & 3 & 3 \\
 6 & 0 & 5 & 6 & 1 & 2 \\
 7 & 4 & 1 & 7 & 0 & 1 \\
\end{array}
\]

(a) The function \( h \) is defined by \( h(x) = f(g(x)) \). Use the chain rule to find \( h'(3) \).

**Solution:** By the chain rule, \( h'(3) = f'(g(3)) \cdot g'(3) = f'(2) \cdot g'(3) = 4 \cdot 6 = 24 \).
(b) The function $k$ is defined by $k(x) = f(x) \cdot g(x)$. Use the product rule to find $k'(1)$.

Solution: $k'(1) = f'(1) \cdot g(1) + f(1) \cdot g'(1) = 3 \cdot 7 + 2 \cdot 3 = 27$.

(c) The function $H$ is defined by $H(x) = f(f(x))$. Use the chain rule to find $H'(2)$.

Solution: $H'(2) = f'(f(2)) \cdot f'(2) = f'(5) \cdot f'(2) = 4 \cdot 4 = 16$.

(d) Let $Q(x) = f(f(x) - g(x))$. Find $Q'(5)$.

Solution: $Q'(5) = f'(f(5) - g(5)) \cdot (f'(5) - g'(5)) = f'(6-3) \cdot (4-3) = 2$. 
9. (10 points) A radioactive substance has a half-life of 27 years. Find an expression for the amount of the substance at time $t$ if 20 grams were present initially.

**Solution:** $Q(t) = Ae^{-kt}$. Since the half-life is 27 years, it follows that $.5 = e^{-27k}$, which can be solved to give $k \approx 0.0025672$. Thus $Q(t) = 20e^{-0.0025672t}$.

10. (10 points) If $h = g \circ f$ and $f(1) = 2$, $g'(2) = 5$, $f'(1) = -3$ find $h'(1)$.

**Solution:** $h'(1) = g'(f(1)) \cdot f'(1) = g'(2) \cdot f'(1) = -15$.

11. (15 points) Let $f(x) = x^4 + 2x^3 - 6x^2 + x - 5$.

(a) Find the interval(s) where $f$ is concave upward.

**Solution:** $f'(x) = 4x^3 + 6x^2 - 12x + 1$ and $f''(x) = 12x^2 + 12x - 12$, which has two zeros, $x = (-1 \pm \sqrt{5})/2$. So $f''$ is positive over the intervals $(-\infty, (-1 - \sqrt{5})/2$ and $(-1 + \sqrt{5})/2, \infty)$.

(b) Find the inflection points of $f$, if there are any.

**Solution:** There are two inflection points, $(-1 + \sqrt{5})/2, -6.0556)$ and $(-1 - \sqrt{5})/2, f(-1 - \sqrt{5})/2) = (-1 - \sqrt{5})/2, -23.944)$
12. (20 points) A ball is thrown upwards from the top of a building that is 200 feet tall. The position of the ball at time $t$ is given by $s(t) = -16t^2 + 36t + 200$, where $s(t)$ is measured in feet and $t$ is measured in seconds.

(a) What is the velocity of the ball at time $t = 0$?
\begin{itemize}
  \item \textbf{Solution:} $s'(t) = -32t + 36$ and $s'(0) = 36$.
\end{itemize}

(b) What is the velocity of the ball at time $t = 1$?
\begin{itemize}
  \item \textbf{Solution:} $s'(1) = -32 \cdot 1 + 36 = 4$.
\end{itemize}

(c) How many seconds elapse before the ball hits the ground?
\begin{itemize}
  \item \textbf{Solution:} Solve $-16t^2 + 36t + 200 = 0$ to get $t \approx 4.83$.
\end{itemize}

(d) What is the speed of the ball when it hits the ground?
\begin{itemize}
  \item \textbf{Solution:} $s'(4.83) \approx -118.72$.
\end{itemize}

(e) What is the acceleration of the ball at the time it hits the ground?
\begin{itemize}
  \item \textbf{Solution:} $a(t) = v'(t) = s''(t) = -32 \text{ ft/sec}^2$.
\end{itemize}
13. (20 points)

(a) Let \( f(x) = 2x^2 \) and compute the Riemann sum of \( f \) over the interval \([1, 9]\) using four subintervals of equal length \( (n = 4) \) and choosing the representative point in each subinterval to be the midpoint of the subinterval.

**Solution:** The endpoints of the intervals are 2, 4, 6, 8 and the sum in question is \( f(2) \cdot (3 - 1) + f(4) \cdot (5 - 3) + f(6) \cdot (7 - 5) + f(8) \cdot (9 - 7) = 2(8 + 32 + 72 + 128) = 480 \).

(b) Compute

\[
\int_{1}^{9} 2x^2 \, dx
\]

and compare this value with the one in part a.

**Solution:** \( \int_{1}^{9} 2x^2 \, dx = \frac{2x^3}{3} \bigg|_{1}^{9} = \frac{2 \cdot 9^3}{3} - \frac{2 \cdot 1^3}{3} = 485\frac{1}{3} \).

14. (10 points) Find an equation for the line tangent to the graph of \( f(x) = x \ln(x) - x \) at the point \((1, f(1))\).

**Solution:** \( f'(x) = \ln(x) + x \cdot \frac{1}{x} - 1 \) so \( f'(1) = 0 + 1 - 1 = 0 \), and since \( f(1) = -1 \), it follows that the tangent line has the equation \( y = -1 \).
15. (10 points) Evaluate $\int 3x^2 \sqrt{x^3 + 1} \, dx$

**Solution:** Let $u = x^3 + 1$. Then $du = 3x^2 \, dx$ and $\int 3x^2 \sqrt{x^3 + 1} \, dx = \frac{2}{3}(x^3 + 1)^{3/2} + C$.

16. (10 points) Evaluate $\int_1^3 x^3 \cdot (x^4 - 2)^2 \, dx$

**Solution:** Let $u = x^4 - 2$. Then $du = 4x^3 \, dx$ and $\int_1^3 x^3 \cdot (x^4 - 2)^2 \, dx = \frac{1}{4}(x^4 - 2)^3|_1^3 = \frac{79^3}{12} - \frac{1}{12} = 41086.5$.

17. (10 points) Evaluate $\int_0^4 2xe^{x^2} \, dx$

**Solution:** $\int_0^4 2xe^{x^2} \, dx = e^{x^2}|_0^4 = e^{16} - e^0 = 8886110.5 - 1 = 8886109.5$. 