1. (8 points) Consider the function $f$ defined by:

$$f(x) = \begin{cases} 
2x^2 - 3 & \text{if } x < 0 \\
5x^2 - 3x & \text{if } x \geq 0
\end{cases}$$

Find an equation for the line that is tangent to the graph of $f$ at the point $(3, f(3))$.

**Solution:** Since $f'(x) = 10x - 3$ for $x > 0$, $f'(3) = 27$. Also, $f(3) = 45 - 9 = 36$, so the line is $y = 36 = 27(x - 3)$ or $y = 27x - 45$.

2. (8 points) Again referring to the function $f$ defined in problem 1, what is the slope of the line joining the points $(-2, f(-2))$ and $(2, f(2))$?

**Solution:** The two points are $(-2, 5)$ and $(2, 14)$, so the slope is $\frac{14 - 5}{2 + 2} = \frac{9}{4}$.

3. (8 points) What is the distance between the point $(6, 8)$ and the midpoint of the segment joining the points $(2, 3)$ and $(10, -7)$?

**Solution:** The midpoint of the segment is $(6, -2)$, so the distance is $d = \sqrt{0^2 + 10^2} = 10$.

4. (8 points) Find an equation for each horizontal asymptote of $r(x)$?

$$r(x) = \frac{(x + 4)(x^2 - 1)(3x^2 - 4)}{(x^2 + x - 12)(x - 1)^4}$$

**Solution:** The degree of the denominator is greater than that of the numerator, so the horizontal asymptote is $y = 0$.

5. (8 points) Referring again to the function $r(x)$ in the previous problem, find an equation for each vertical asymptote of $r(x)$?

**Solution:** The quadratic in the denominator factors, $x^2 + x - 12 = (x + 4)(x - 3)$. The $x + 4$ factor in the denominator cancels with the $x + 4$ factor in the numerator, leaving just the factors $x - 1$ and $x - 3$, so the vertical asymptotes are $x = 1$ and $x = 3$. 
6. (8 points) The line tangent to the graph of a function $f$ at the point $(1,5)$ on the graph also goes through the point $(0,8)$. What is $f'(1)$?

**Solution:** The slope of the line through $(1,5)$ and $(0,8)$ is $-3$.

7. (8 points) What is the slope of the tangent line to the graph of $f(x) = e^{2x}$ at the point $(1,e^2)$?

**Solution:** The derivative is $f'(x) = 2e^{2x}$ whose value of at $x = 1$ is $f'(1) = 2e^2$.

8. (10 points) Let $g(x) = (2x - 3)^2(x + 1)^2$. Find $g'(x)$ and the critical points of $g$. Express $g'$ in factored form.

**Solution:** Use the product rule to compute $g'$: $g'(x) = 2(2x - 3)^{2-1} \cdot 2(x + 1) + 2(x + 1)^{1-1} \cdot (2x - 3)^2$. Factor out the common terms to get $g'(x) = (2x - 3)(x + 1)(4x + 2x - 3) = (2x - 3)(x + 1)(8x - 2) = 2(2x - 3)(x + 1)(4x - 1)$. So the stationary points are $x = 3/2$, $x = -1$, and $x = 1/4$.

9. (10 points) Let $f(x)$ be the function defined on $[-3,4]$ by the equation $f(x) = x^3 - 9x + 4$. Find the absolute maximum and absolute minimum of $f$ and the locations where those extrema occur.

**Solution:** The derivative is $f'(x) = 3x^2 - 9$, so the stationary points are $x = \pm \sqrt{3}$. To find the absolute max and min we have to compare the values of $f$ at the stationary points and at the endpoints, $-3$ and $4$. Note that $f(-\sqrt{3}) = -3\sqrt{3}$. The graph is shown below.

![Graph of f(x) = x^3 - 9x + 4]
10. (15 points) Suppose $f''(x) = (x - 5)(2x + 3)(x - 3)(2x + 9)$. Find the intervals over which $f$ is concave upwards. Note that the second derivative has already been found for you.

**Solution:** The four critical points are $x = -9/2, -3/2, 3,$ and $5$. Use the test interval technique to solve the inequality $f''(x) > 0$. You could use the test points $x = -5, -2, 0, 4,$ and $6$. You find that $f''(-5), f''(0),$ and $f''(6)$ are positive. Thus $f(x)$ is concave upwards on each of the intervals $(-\infty, -9/2), (-3/2, 3), (5, \infty)$.

11. (10 points) The graph of a function $G(x)$ is shown below. Sketch the graph of the derivative function $G'(x)$ on the same coordinate system.

**Solution:** The function $G(x)$ has a horizontal tangent line at roughly $x = 1/2$, so the function $G'(x)$ must have a zero there. Since the tangent line at $x = 3$ is vertical, $G'(x)$ has a vertical asymptote at $3$. Below are sketches of both $G(x)$ and $G'(x)$ on the same set of axes.
12. (40 points) Compute the following antiderivatives.

(a) \( \int 6x^3 - 5x - 1 \, dx \)

**Solution:** use the power rule to antidifferentiate: \( \int 6x^3 - 5x - 1 \, dx = 3x^3/2 + 5x^2/2 - x + C. \)

(b) \( \int 6x^{\frac{3}{2}} + x^{-\frac{1}{2}} \, dx \)

**Solution:** Again use the power rule to antidifferentiate: \( \int 6x^{\frac{3}{2}} + x^{-\frac{1}{2}} \, dx = 6 \cdot 2/5 \cdot x^{5/2} + 2x^{1/2} + C. \)

(c) \( \int \frac{3x^3 + 2x - 1}{x} \, dx \)

**Solution:** Simplify by dividing first and then antidifferentiate term by term to get \( \int \frac{3x^3 + 2x - 1}{x} \, dx = \int 3x^2 + 2 - 1/x \, dx = x^3 + 2x - \ln |x| + C. \)

(d) \( \int \frac{2x + 1}{x^2 + x - 3} \, dx \)

**Solution:** = \( \ln |x^2 + x - 3| + C. \)
13. (10 points) The graph of $f'(x)$ is given below. Suppose $f(0) = 0$. Sketch the graph of $f(x)$ on the same coordinate system. Notice that $f'$ is defined only on the interval $[-2, 5]$.

Solution: Notice that the graph of $f'$ seems to be a parabola that open upwards and has two zeros, at $x = -1$ and $x = 4$. We might therefore guess that $f'$ has a symbolic representation that is close to $-(x + 1)(x - 4)$. This doesn’t quite do it but it gets us in the ballpark. It shows that we should try $f$ as a cubic polynomial. Antidifferentiating $-(x + 1)(x - 4)$ results in $f(x) = -x^3/3 + 3x^2/2 - 4x$ which is increasing precisely where $f'$ is positive, decreasing exactly where $f'$ is negative and satisfies $f(0) = 0$. The only problem is $f'$ is much flatter than $-(x + 1)(x - 4)$, so $f$ must be flatter as well. The point is that the behavior of $f'$ near $-1$ shows that $f$ must have a relative minimum at $x = -1$ and the behavior of $f'$ near $4$ implies that $f$ has a relative maximum at $4$. The graphs of both $f'$ and $f$ are shown below.
14. (10 points) Find a function $G(x)$ that satisfies $G'(x) = 3x^2 - 7x$ and $G(4) = 9$.

**Solution:** Antidifferentiating $G'$ gives $G(x) = x^3 - \frac{7}{2}x^2 + C$ where $C$ is a constant that has to be determined. Since $G(4) = 9 = 4^3 - \frac{7}{2} \cdot 4^2 + C$, it follows that $C = 9 - 8 = 1$, and that $G(x) = x^3 - \frac{7}{2}x^2 + 1$.

15. (40 points) Compute each of the following derivatives.

(a) $\frac{d}{dx} \sqrt{x^4 + 1}$

**Solution:** By the chain rule, $\frac{d}{dx} \sqrt{x^4 + 1} = 4x^3 \cdot \frac{1}{2} (x^4 + 1)^{-1/2} = 2x^3 / \sqrt{x^4 + 1}$.

(b) $\frac{d}{dx} [\ln(2x + 1)]^3$

**Solution:** Again by the chain rule, $\frac{d}{dx} [\ln(2x + 1)]^3 = 3 \cdot \ln(2x + 1) \cdot \frac{2}{2x + 1} = \frac{6(\ln(2x+1))^2}{2x+1}$.

(c) $\frac{d}{dx} x^2 e^{2x+1}$

**Solution:** By the product rule, $\frac{d}{dx} x^2 e^{2x+1} = 2xe^{2x+1} + 2x^2 e^{2x+1} = 2xe^{2x+1}(1 + x)$. 
(d) \[
\frac{d}{dx} \frac{2x^2 - 3}{2x + 3}
\]

Solution: By the quotient rule, \[
\frac{d}{dx} \frac{2x^2 - 3}{2x + 3} = \frac{4x(2x + 3) - 2(2x^2 - 3)}{(2x + 3)^2} = \frac{2x^2 + 6x + 3}{(2x + 3)^2},
\]
the numerator of which does not factor.

16. (20 points) Compute the following integrals.

(a) \[
\int_0^2 2xe^{-x^2} \, dx
\]

Solution: \[
\int_0^2 2xe^{-x^2} \, dx = -e^{-x^2}\bigg|_0^2 = -e^{-4} - (-1) = 1 - 1/e^4 \approx .9816.
\]

(b) \[
\int_0^5 (2x - 1)\sqrt{x^2 - x + 5} \, dx
\]

Solution: By substitution, let \( u = x^2 - x + 5 \). Then \( du = 2x - 1 \, dx \), and we have \[
\int_0^5 (2x - 1)\sqrt{x^2 - x + 5} \, dx = \int \sqrt{u} \, du = \frac{2}{3}u^{3/2} = \frac{2}{3}(x^2 - x + 5)^{3/2}\bigg|_0^5 = \frac{2}{3}(125 - 5\sqrt{5}) \approx 75.88.
\]

17. (10 points) Find the area of the region bounded by \( y = x^{3/2} \), the \( x \)-axis, and the lines \( x = 0 \) and \( x = 4 \).

Solution: The integral is \[
\int_0^4 x^{3/2} \, dx = \frac{2}{5}x^{5/2}\bigg|_0^4 = \frac{64}{5} = 12.8.
\]

18. (15 points) Find the area \( A \) of the region caught between the graphs of the functions \( f(x) = -x^2 + 4x \) and \( g(x) = -2x + 5 \).

Solution: First find the two points of intersection of the graphs by solving \(-x^2 + 4x = -2x + 5\) to get \( x = 1 \) and \( x = 5 \). Then integrate the difference \( f(x) - g(x) \) from \( x = 1 \) to \( x = 5 \). Thus \( A = \int_1^5 (-x^2 + 4x + 2x - 5) \, dx = -x^3/3 + 3x^2 - 5x\bigg|_1^5 = 10\frac{2}{3} \).