May 6, 2002

As usual, show all your work. If you used a calculator, explain in detail how you used it to solve the problem. Part A is worth 120 points and part B 130 points.

1. (40 points) Find the following antiderivatives.

(a) \[ \int 6x^3 - 5x - 1 \, dx \]

**Solution:** \[ \frac{3}{2} x^4 - \frac{5}{2} x^2 - x + C. \]

(b) \[ \int 6x^2 + x^{-\frac{1}{2}} \, dx \]

**Solution:** \[ 6 \cdot \frac{2}{5} \cdot x^{5/2} + \frac{2}{1} \cdot x^{1/2} + C. \]

(c) \[ \int \frac{3x^3 + 2x - 1}{x} \, dx \]

**Solution:** \[ \int 3x^2 + 2 - \frac{1}{x} \, dx = 3x^2/2 + 2x - \ln|x| + C. \]

(d) \[ \int \frac{2x + 1}{x^2 + x - 3} \, dx \]

**Solution:** By substitution, \( u = x^2 + x - 3 \), \[ \int \frac{2x + 1}{x^2 + x - 3} \, dx = \ln|x^2 + x - 3| + C. \]

(e) \[ \int 5x^4 (x^5 + 2)^7 \, dx \]

**Solution:** By substitution with \( u = x^5 + 2 \), \[ \int 5x^4 (x^5 + 2)^7 \, dx = \frac{(x^5 + 2)^8}{8} + C. \]

2. (20 points) Compute the following definite integrals.

(a) \[ \int_0^2 2xe^{-x^2} \, dx \]

**Solution:** \[ -e^{-x^2}\big|_0^2 = 1 - e^{-4} \approx 0.9816. \]

(b) \[ \int_0^5 (2x - 1)\sqrt{x^2 - x + 5} \, dx \]

**Solution:** \[ 2/3(x^2 - x + 5)^{3/2}|_0^5 = \frac{10}{3}(25 - \sqrt{5}) \approx 75.8798. \]

3. (15 points) Find a function \( G(x) \) whose derivative is \( 3x^2 - 7 \) and for which \( G(4) = 9 \).

**Solution:** \( G(x) = x^3 - 7x + C \) for some constant \( C \). But since \( G(4) = 9 = 4^3 - 28 + C \), it follows that \( C = -36 + 9 = -27 \) and \( G(x) = x^3 - 7x - 27 \).

4. (15 points) Find the area of the region bounded by \( y = x^{3/2} \), the \( x \)-axis, and the lines \( x = 0 \) and \( x = 4 \).

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Solution: The area is given by 
\[
\int_0^4 x^{3/2} \, dx = \left. \frac{2}{5} x^{5/2} \right|_0^4 = \frac{2}{5} (4^{5/2} - 0^{5/2}) = \frac{2}{5} \cdot 32 = 64.
\]

5. (30 points) An object is thrown upwards from the top of a 400 feet high building, after which its path is governed entirely by gravity. The acceleration due to gravity is given by \( a(t) = -32 \), where \( t \) is measured in seconds and \( a(t) \) is measured in \( \text{ft/sec}^2 \). Recall that \( a(t) = v'(t) = s''(t) \), where \( v(t) \) denotes the velocity of the object (negative when its moving towards the earth), and \( s(t) \) is the position of the object at time \( t \). The equation above simply says that the velocity is an antiderivative of acceleration, and that position is an antiderivative of velocity. Suppose the initial velocity is 80 feet per second; ie, \( v(0) = 80 \). Answer the following questions about the path of the object.

(a) Compute the function \( v(t) \).

Solution: \( v(t) = -32t + C = -32 + 80 \) since \( V(0) = 80 \).

(b) Compute the function \( s(t) \).

Solution: \( s(t) = -16t^2 + 80t + C = -16t^2 + 80t + 400 \) because \( s(0) = 400 \).

(c) At what time does the object hit the ground?

Solution: Solve the equation \( 16t^2 - 80t - 400 = 0 \) to find that \( t = \frac{5 \pm \sqrt{25 + 100}}{2} \), only one of which is positive. Thus, \( t = \frac{5 + \sqrt{75}}{2} \approx 8.09 \).

(d) At what time does the object reach its maximum height?

Solution: \( v'(t) = -32t + 80 = 0 \) happens at \( t = 2.5 \)

(e) What is its maximum height?

Solution: \( s(2.5) = -16(2.5)^2 + 80(2.5) + 400 = 574.70 \) ft.
Part B

1. (5 points) The slope of the line that contains the points \((-1, y)\) and \((4, -12)\) is \(-3\)? What is \(y\)?

\textbf{Solution:} Solve the equation \(\frac{y + 12}{-1 - 4} = -3\) to get \(y = 3\).

2. (5 points) What is the slope of the line perpendicular to the line \(2y + x = 4\)?

\textbf{Solution:} The slope of the given line is \(-\frac{1}{2}\) so the slope of its perpendicular is 2.

3. (10 points) Over what intervals is the second derivative of \(g(x) = x^4 - 6x^3 + 12x^2 + 2x + 2\) negative?

\textbf{Solution:} Since \(g'(x) = 4x^3 - 18x^2 + 24x + 2\), it follows that \(g''(x) = 12x^2 - 36x + 24 = 12(x - 1)(x - 2)\). Use the Test Interval Technique to determine that \(g''\) is negative on the interval \((1, 2)\).

4. (15 points) Construct a cubic polynomial \(f(x)\) that has zeros at \(x = -2, 1, 3\) and satisfies \(f(0) = -12\).

\textbf{Solution:} Since the zeros are \(-2, 1,\) and \(3\), the function must be \(f(x) = a(x + 2)(x - 1)(x - 3)\) for some constant \(a\). Then \(f(0) = a(2)(-1)(-3) = 6a = -12\), so \(a = -2\), and \(f(x) = -2(x + 2)(x - 1)(x - 3)\).

5. (10 points) Let \(g(x)\) be defined as follows: Let

\[g(x) = \begin{cases} e^x & \text{if } x \leq 1 \\ \ln(x) & \text{if } x > 1 \end{cases}\]

Find an equation for the line tangent to the graph of \(g(x)\) at the point \((3, g(3))\).

\textbf{Solution:} First note that \(g'(3) = 1/3\) and that \(g(3) = \ln(3)\) so the line is \(y - \ln(3) = \frac{1}{3}(x - 3)\), which simplifies to \(y = x/3 - 1 + \ln(3)\).

6. (20 points) Compute each of the following derivatives.

\textbf{(a)} \(\frac{d}{dx} \frac{\sqrt{x^2 - 3}}{2x}\)

\textbf{Solution:} By the quotient rule, \(\frac{d}{dx} \frac{\sqrt{x^2 - 3}}{2x} = \frac{x^2(x^2 - 3)^{-1/2} - 2(x^2 - 3)^{1/2}}{4x^2}\), which simplifies slightly.

\textbf{(b)} \(\frac{d}{dx} e^{x + \ln(x)}\)

\textbf{Solution:} By the chain rule, \(\frac{d}{dx} e^{x + \ln(x)} = e^{x + \ln(x)}(1 + \frac{1}{x})\).

\textbf{(c)} \(\frac{d}{dx} \ln(x^2 + e^{2x})\)

\textbf{Solution:} By the chain rule, \(\frac{d}{dx} \ln(x^2 + e^{2x}) = \frac{2x + 2e^{2x}}{x^2 + e^{2x}}\).
7. (10 points) Calculate the doubling time for a 7% investment compounded continuously.

**Solution:** Solve $2P = Pe^{rt}$ where $r = 0.07$ to get $2 = e^{0.07t}$ or $t = \frac{\ln(2)}{0.07} \approx 9.902$ years.

8. (30 points) Suppose $u(x)$ is a function whose derivative $u'(x) = (2x + 1)^2(x - 2)^2$. Recall that theorem B tells you the intervals over which $u(x)$ is concave upwards based on $u''(x)$.

(a) Compute $u''(x)$.

**Solution:**

\[ u''(x) = 2(2x + 1)(2x - 2)^2 + (2x + 1)^2 \cdot 2(x - 2) = 2(2x + 1)(x - 2)(4x - 3). \]

(b) Find the three zeros of $u''(x)$.

**Solution:** $x = -1/2$, $x = 2$, and $x = 3/4$.

(c) Use the Test Interval Technique to find the intervals over which $u(x)$ is concave up.

**Solution:** $u$ is concave up over $(-1/2, 3/4)$ and over $(2, \infty)$.

9. (5 points) What is $\lim_{h \to 0} \frac{1 - \sqrt{1 + 2h}}{h}$?

**Solution:** Rationalize the numerator to get $\lim_{h \to 0} \frac{-2}{1 + \sqrt{1 + 2h}} = -1$.

10. (20 points) Find the absolute maximum and absolute minimum of the function $h(x) = \sqrt{x^2 + 6x + 25}$ over the interval $[-5, 5]$.

**Solution:** Differentiate to get $h'(x) = (2x + 6)(x^2 + 6x + 25)^{-1/2}$ so the only critical point is $x = -3$. Checking endpoints, we have $h(-5) = \sqrt{20}$, $h(-3) = \sqrt{16}$ and $h(5) = 4\sqrt{5}$. So the minimum value is 4 which occurs at $x = -3$ and the maximum value is $4\sqrt{5}$ which occurs at $x = 5$. 