Math 1120 Calculus Test 2

October 7, 1999 Name

The first five problems counts 6 points each and the others count as marked.

Multiple choice section. Circle the correct choice. You do not need to show your work on these problems.

1. Consider the function \( f \) defined by:

\[
\begin{align*}
    f(x) &= \begin{cases} 
        |x + 2| & \text{if } x \leq 0 \\
        5 - x^2 & \text{if } x > 0
    \end{cases}
\end{align*}
\]

Find the three solutions to \( f(x) = 1 \) and compute their sum.

(A) \(-4\) \hspace{1cm} (B) \(-2\) \hspace{1cm} (C) \(0\) \hspace{1cm} (D) \(2\) \hspace{1cm} (E) \(6\)

2. Let \( f(x) = \frac{1}{x} \). What is the value of \( \frac{f(x + 2) - f(x)}{2} \)?

(A) \(-\frac{1}{x(x + 2)}\) \hspace{1cm} (B) \(\frac{1}{x(x + 2)}\) \hspace{1cm} (C) \(\frac{x}{x + 2}\) \hspace{1cm} (D) \(-\frac{x}{x + 2}\) \hspace{1cm} (E) \(x + 2\)

3. Let \( f(x) = \sqrt{2x} \). What is the value of \( f(x + 1) - f(x) \) in terms of \( x \)?

(A) \(\frac{2}{\sqrt{2x + 2} + \sqrt{2x}}\) \hspace{1cm} (B) \(\frac{2}{\sqrt{2x + 1} + \sqrt{2x}}\) \hspace{1cm} (C) \(\frac{1}{\sqrt{2x + 1}}\)

(D) \(\sqrt{2x + 2}\) \hspace{1cm} (E) \(\sqrt{2x + 2} - x\)

4. Suppose the point \((2, 5)\) belongs to the graph of a function \(g\) and \(g'(2) = 4\). What is the \(y\)-intercept of the line tangent to the graph of \(g\) at the point \((2, 5)\)?

(A) \(-8\) \hspace{1cm} (B) \(-3\) \hspace{1cm} (C) \(3\) \hspace{1cm} (D) \(8\) \hspace{1cm} (E) \(13\)

5. The line tangent to the graph of a function \(h\) at the point \((3, 7)\) has a \(y\)-intercept of 10. What is \(h'(3)\)?

(A) \(-7\) \hspace{1cm} (B) \(-4\) \hspace{1cm} (C) \(-1\) \hspace{1cm} (D) \(1\) \hspace{1cm} (E) \(17/3\)
6. (20 points) Let

\[ f(x) = \begin{cases} 
2x - 3 & \text{if } x \leq 4 \\
6 - x & \text{if } x > 4 
\end{cases} \]

and let \( g(x) = 2x \).

(a) Compute each of the following

i. \( f \circ g(1) \)
\[
\begin{array}{l}
\text{if } x \leq 4 \\
\text{if } x > 4 
\end{array}
\]

\( \frown g(1) = f(2) = 1 \)

ii. \( f \circ g(2) \)
\[
\begin{array}{l}
\text{if } x \leq 4 \\
\text{if } x > 4 
\end{array}
\]

\( \frown g(2) = f(4) = 5 \)

iii. \( f \circ g(3) \)
\[
\begin{array}{l}
\text{if } x \leq 4 \\
\text{if } x > 4 
\end{array}
\]

\( \frown g(3) = f(6) = 0 \)

iv. \( f \circ g(3.5) \)
\[
\begin{array}{l}
\text{if } x \leq 4 \\
\text{if } x > 4 
\end{array}
\]

\( \frown g(3.5) = f(7) = -1 \)

(b) Find a symbolic representation of the composition \( f \circ g(x) \), and simplify the representation.

\[ f \circ g(x) = \begin{cases} 
2(2x) - 3 & \text{if } 2x \leq 4 \\
6 - 2x & \text{if } 2x > 4 
\end{cases} \]

which is the same as

\[ f \circ g(x) = \begin{cases} 
4x - 3 & \text{if } x \leq 2 \\
6 - 2x & \text{if } x > 2 
\end{cases} \]
7. (25 points) Compute the limits requested.

(a) \( \lim_{h \to 0} \frac{\sqrt{2 + h} - \sqrt{2}}{h} \)

Rationalize the numerator to get \( \frac{(2+h)-2}{h(\sqrt{2+h}+\sqrt{2})} = \frac{h}{h(\sqrt{2+h}+\sqrt{2})} \)

so the limit is just the value of the last expression at \( h = 0 \), which is \( \frac{1}{2\sqrt{2}} \).

(b) \( \lim_{x \to 3} \frac{x - 3}{x^3 - 27} \)

Factor the denominator to get \( \frac{x-3}{(x-3)(x^2+3x+9)} \). The \( (x-3) \) factors can be removed to give \( \lim_{x \to 3} \frac{1}{x^2 + 3x + 9} = 1/27 \).

(c) \( \lim_{h \to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} \)

Find a common denominator and simplify to get \( \lim_{h \to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} = \lim_{h \to 0} \frac{3-(3+h)}{h\cdot3(3+h)} \) which is just \( \lim_{h \to 0} \frac{-1}{3(3+h)} = -1/9 \).

(d) \( \lim_{x \to \infty} \frac{2x^3 - 2x^2 + 7}{4x^3 - 10x^2 + x - 27} \)

Since the degrees of the numerator and denominator are the same, take the ratio of the coefficients of these highest power terms, \( 2x^3 \) and \( 4x^3 \) to get the result \( \lim = 1/2 \).

(e) \( \lim_{x \to -\infty} \frac{|x| - 3}{3x + 5} \)

Divide both numerator and denominator by \( x \) to get \( \lim_{x \to -\infty} \frac{|x| - 3}{3x + 5} = \lim_{x \to -\infty} \frac{|x|/x - 3/x}{3x/x + 5/x} \). This reduces to \( \lim_{x \to -\infty} \frac{|x|/x}{3} \) whose value, for negative values of \( x \), is \(-1/3\).
8. (25 points) Find the following derivatives.

(a) \( \frac{d}{dx} \sqrt{2x^3 - 5x + 7} \)
\[
\frac{d}{dx} \sqrt{2x^3 - 5x + 7} = \frac{1}{2}(2x^3 - 5x + 7)^{-\frac{1}{2}} \cdot (6x^2 - 5)
\]

(b) \( \frac{d}{dx} (2x - 1) \cdot (3x^2 + 4x) \)
\[
\frac{d}{dx} (2x - 1) \cdot (3x^2 + 4x) = (2x - 1)(6x + 4) + (3x^2 + 4x) \cdot 2 \\
= 12x^2 + 8x - 6x - 4 + 6x^2 + 8x \\
= 18x^2 + 10x - 4
\]
Alternatively, multiply the two factors and differentiate the result.

(c) \( \frac{d}{dx} \frac{2x^2 - 1}{3x + 2} \)
\[
\frac{d}{dx} \frac{2x^2 - 1}{3x + 2} = \frac{4x(3x + 2) - 3(2x^2 - 1)}{(3x + 2)^2} \\
= \frac{12x^2 + 8x - 6x^2 + 3}{(3x + 2)^2} \\
= \frac{6x^2 + 8x + 3}{(3x + 2)^2}
\]

(d) \( \frac{d}{dx} \sqrt{x^2 - 2x + 1} \)
\[
\frac{d}{dx} \sqrt{x^2 - 2x + 1} = \frac{d}{dx} \sqrt{(x - 1)^2} = \frac{d}{dx}|x - 1| = \frac{|x - 1|}{x - 1}
\]

(e) \( \frac{d}{dx} (x^3 + 3x^2 + 3x + 1)^{1/3} \)
\[
\frac{d}{dx} (x^3 + 3x^2 + 3x + 1)^{1/3} = \frac{d}{dx} ((x + 1)^3)^{1/3} = \frac{d}{dx} (x + 1) = 1
\]
9. (20 points) Let \( f(x) = \frac{1}{x} + x \).

(a) Compute \( f(3.1) \) \( f(3.1) = \frac{1}{3.1} + 3.1 \approx 3.42 \)

(b) Compute \( f(3 + h) \) \( f(3 + h) = \frac{1}{3+h} + 3 + h \approx 3.42 \)

(c) Compute \( \frac{f(3+h) - f(3)}{h} \) and simplify, assuming \( h \neq 0 \).

\[
\frac{f(3+h) - f(3)}{h} = \frac{\frac{1}{3+h} + 3 + h - \left( \frac{1}{3} + 3 \right)}{h}
= \frac{\frac{1}{3+h} - \frac{1}{3} + h}{h}
= \frac{3 - (3 + h)}{h \cdot 3 \cdot (3 + h)} + 1
= - \frac{h}{3h(3 + h)} + 1
= - \frac{1}{3(3 + h)} + 1
\]

(d) Take the limit of the expression in (c) as \( h \) approaches 0 to find \( f'(3) \).

The limit as \( h \) approaches 0 is \( -\frac{1}{9} + 1 = \frac{8}{9} \).

(e) What is the slope of the line tangent to \( f \) at the point \( (3, \frac{3}{3}) \).

Its the number we just calculated, \( \frac{8}{9} \).

(f) Find an equation for the line tangent to the graph of \( f \) at the point \( (3, \frac{3}{3}) \).

The equation is \( y - \frac{3}{3} = \frac{8}{9}(x - 3) \), which simplifies to \( y = \frac{8}{9}x + \frac{2}{3} \).