April 11, 2003

The first 6 problems count 5 points each. Problems 6 through 9 count as marked. In the multiple choice section, circle the correct choice (or choices). The total number of points available is 120.

Each of the next few items are true-false. To get full credit you must give a valid reason for your answer. Circle either True or False, and give your reason in the space provided. Generally, 2 points for the right t/f value and 3 points for the right reason.

1. True or false. If \( f''(x) < 0 \) on the interval \((a, c)\) and \( f''(x) > 0 \) on the interval \((c, b)\), then the point \((c, f(c))\) is a point of inflection of \( f \).

   **Solution:** True. This is basically the definition of inflection point.

2. True or false. If \( f'(c) = 0 \), then \( f \) has a relative maximum or a relative minimum at \( x = c \).

   **Solution:** False. The function can have neither a max nor a min at a stationary point. Look at \( f(x) = x^3 \) and \( x = 0 \).

3. True or false. If \( f \) has a relative maximum at \( x = c \), then \( f'(c) = 0 \).

   **Solution:** False. All we can tell is that \( c \) is a critical point. It might be a singular point.

4. True or false. If \( f'(c) = 0 \) and \( f''(c) < 0 \), then \( f \) has a relative maximum at \( x = c \).

   **Solution:** True. This is just the second derivative test.

5. True or false. If \( h(x) = \sqrt{x^2 - 4} \), then \( h'(x) = \frac{1}{2}(x^2 - 4)^{-1/2} \).

   **Solution:** False. Look up the chain rule.

6. True or false. The function \( g(x) = (x - 1)^{2/3} \) has a singular point at \( x = 1 \).

   **Solution:** True. Since \( g'(x) = 2(x - 1)^{-1/3}/3 = 2/3(x - 1)^{-1/3} \), you can see that \( g'(1) \) is not a number. Therefore \( x = 1 \) is a singular point.
On all the following questions, **show your work**.

7. (20 points) Sketch the graph of a function $g(x)$ satisfying the properties shown in the table below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
<th>$g'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Use the coordinate system given.

**Solution:** One possible graph is
8. (20 points) Let \( g(x) = (2x - 4)^2(x + 3)^2 \).

(a) Use the test interval technique (not a graphing calculator) to find the intervals over which \( g \) is increasing.

**Solution:** First, note that \( g'(x) = 2(2x - 4)(x + 3)[2x + 6 + 2x - 4] = 2(2x - 4)(x + 3)(4x + 2) \), so the critical points are \( x = -3, x = -1/2, \) and \( x = 2 \). Using the test interval technique on the intervals \((-\infty, -3), (-3, -1/2), (-1/2, 2) \) and \((2, \infty) \) with test points \(-4, -1, 0, \) and \(3 \), we can see that

<table>
<thead>
<tr>
<th>test point</th>
<th>2x-4</th>
<th>x+3</th>
<th>4x+2</th>
<th>sign of ( g' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>-1</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>0</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

We conclude that \( g(x) \) is increasing on the two intervals \((-3, -1/2) \) and \((2, \infty) \).

(b) Find and classify each critical point as a location of a. a relative maximum, b. a relative minimum, or c. neither a relative max nor a relative min.

**Solution:** We can use the second derivative test on the three critical points of \( g \). Since \( g''(x) = 2 \cdot 2(x + 3)(4x + 2) + 2(2x - 4)(4x + 2 + 4(x + 3)) \), by the product rule, it follows that \( g''(-3) = 200, g''(-1/2) = -100, \) and \( g''(2) = 200 \). The second derivative test now confirms what we felt, that is, there is a relative maximum at \(-1/2 \) and relative minimums at both \(-3 \) and \(2 \).

9. (15 points) Consider the rational function

\[
f(x) = \frac{(x^2 - 4)(2x + 1)}{(3x^2 - 3)(x - 2)}
\]

(a) Find the horizontal asymptote(s).

**Solution:** The coefficient of \( x^3 \) in the numerator is 2 while that in the denominator is 3, so \( y = 2/3 \) is the horizontal asymptote.

(b) Find the vertical asymptotes.

**Solution:** To find the vertical asymptotes, you must first reduce the fraction to lowest terms, which mean cancelling out the common factors,
in this case, just the \( x - 2 \)'s. This results in a denominator that has value 0 only at \( x = \pm 1 \), so these are the two vertical asymptotes.

(c) Compute \( \lim_{x \to \infty} f(x) \).

**Solution:** The limit in question is the same as the horizontal asymptote, \( \frac{2}{3} \).

10. (15 points) Four congruent \( x \times x \) squares from the corners of a cardboard rectangle that measures 16 \( \times \) 12. The sides are then folded upward to form a topless box.

(a) Find the volume \( V \) as a function of \( x \). What is the logical domain?

**Solution:** \( V(x) = (16 - 2x)(12 - 2x)x \).

(b) Compute \( V(0) \), \( V(1) \), \( V(2) \), and \( V(3) \).

**Solution:** \( V(0) = 0, V(1) = 14 \cdot 10 \cdot 1 = 140, V(2)12 \cdot 8 \cdot 2 = 192, \) and \( V(3) = 10 \cdot 6 \cdot 3 = 180. \)

(c) Find \( V'(x) \).

**Solution:**

\[
V'(x) = -2(12 - 2x)x + (16 - 2x)(\frac{d}{dx}(12 - 2x)x)
\]
\[
= -2(12 - 2x)x + (16 - 2x)[(-2)x + 1(12 - 2x)]
\]
\[
= -24x + 4x^2 + (16 - 2x)(-4x + 12)
\]
\[
= 12x^2 - 112x + 192
\]

(d) Use the results from the question above to determine the critical points of \( V \).

**Solution:** Use the quadratic formula to find that the critical point is \( c = \frac{14 - 2\sqrt{13}}{3} \approx 2.263. \) The other root of the quadratic is outside the domain of \( V \).

(e) Find the absolute maximum value of \( V \) and the value of \( x \) where it occurs.

**Solution:** Simply evaluate \( V \) at the two endpoints and the critical point to see that the maximum value of \( V \) is \( V(c) \approx 194.07 \).
11. (20 points) Compute each of the following derivatives.

(a) \( \frac{d}{dx} \sqrt{x^3 + 1} \)
Solution: \( \frac{d}{dx} \sqrt{x^3 + 1} = \frac{3x^2}{2\sqrt{x^3+1}}. \)

(b) \( \frac{d}{dx} (2x^2 + 1)^{10} \)
Solution: \( \frac{d}{dx} (2x^2 + 1)^{10} = 40x(2x^2 + 1)^9. \)

(c) \( \frac{d}{dx} \left( \frac{2x + 1}{x^2 + 1} \right) \)
Solution: \( \frac{d}{dx} \left( \frac{2x + 1}{x^2 + 1} \right) = -\frac{2(x^2 + x - 1)}{(x^2 + 1)^2}. \)

(d) \( \frac{d}{dx} (2x^2 + 1)(3x - 4) \)
Solution: \( \frac{d}{dx} (2x^2 + 1)(3x - 4) = 4x(3x - 4) + 3(2x^2 + 1) = 18x^2 - 16x + 3. \)