1. (15 points) How long does it take an 8% investment to triple if

(a) Compounding takes place monthly?

Solution: To find a value of $t$ for which $3 = 1 \left(1 + \frac{0.08}{12}\right)^{12t}$, take ln of both sides to get $12t \ln (1 + 2/300) = \ln 3$. The value of $t$ satisfying this equation is $t \approx 13.778$ years.

(b) Compounding takes place continuously?

Solution: We need to solve $3 = e^{0.08t}$. This is easy if we take ln of both sides. We get $0.08t = \ln 3$ which yields $t \approx 13.732$ years, just a little less than the answer to (a), as expected.

2. (15 points) Let $f(x) = x^4 + 2x^3 - 12x^2 + x - 5$.

(a) Find the interval(s) where $f$ is concave upward.

Solution: $f'(x) = 4x^3 + 6x^2 - 24x + 1$ and $f''(x) = 12x^2 + 12x - 24$, which has two zeros, $x = -2$ and $x = 1$. So $f''$ is positive over the intervals $(-\infty, -2)$ and $(1, \infty)$.

(b) Find the inflection points of $f$, if there are any.

Solution: There are two inflection points, $(-2, f(-2)) = (-2, -55)$ and $(1, f(1)) = (1, -13)$
3. (15 points) Find the absolute maximum value and the absolute minimum value of the function \( f(x) = x^3 - 6x^2 + 8x + 7 \) on the interval \( 0 \leq x \leq 6 \).

**Solution:** Note that \( f'(x) = 3x^2 - 12x + 8 \), so we get the critical points using the quadratic formula:

\[
x = \frac{12 \pm \sqrt{(-12)^2 - 4 \cdot 3 \cdot 8}}{2 \cdot 3} = 2 \pm \frac{2\sqrt{3}}{3}.
\]

Thus we need to compare the four numbers \( f(0) \), \( f(2 + 2\sqrt{3}/3) \), \( f(2 - 2\sqrt{3}/2) \), and \( f(6) \). These values are \( f(0) = 7 \), \( f(2 + 2\sqrt{3}/3) \approx 3.9207 \), \( f(2 - 2\sqrt{3}/2) \approx 9.2945 \), and \( f(6) = 55 \). So the maximum of 55 occurs at \( x = 6 \) and the minimum of about 3.9207 at the point \( x = 2 + 2\sqrt{3}/3 \).

4. (15 points) Suppose the function \( Q(t) = Q_0e^{-kt} \) satisfies \( Q(5770) = Q_0/2 \).

(a) What is the value of \( k \)?

**Solution:** The number \( k \) satisfies \( Q_0/2 = Q_0e^{-k(5770)} \). That is \( e^{-5770k} = 0.5 \). Take ln of both sides to get \(-5770k = \ln 0.5 \approx -0.693\) so \( k \approx 1.2019 \times 10^{-4} = 0.00012019 \).

(b) For what value of \( t \) is it true that \( Q(t) = Q_0/4 \)?

**Solution:** Since the half-life is 5770, the quarter-life is 11540.

(c) Find \( Q'(t) \).

**Solution:** \( Q'(t) = -kQ_0e^{-kt} \).

(d) What is the rate of growth of \( Q(t) \) at \( t = 11540 \)?

**Solution:** \( Q'(11540) = -kQ(11540) = -kQ_0/4 \approx -3.0032 \times 10^{-5}Q_0 = -0.000030032Q_0 \).
5. (15 points) Find the interval(s) where \( f(x) = (x - 4)(x^2 - 1)(x + 3) \) is positive.

**Solution:** The branch points are \( x = 4, 1, -1, \) and \(-3\). I picked test points \(-4, -2, 0,\) and \(5,\) and found that \( f(-4) > 0, \ f(-2) < 0, \ f(0) > 0, \ f(2) < 0, \) and \( f(5) > 0. \) Therefore, the function \( f \) is positive on \((-\infty, -3), (-1, 1),\) and \((4, \infty)).\)

6. (15 points) Compute each of the following derivatives.

(a) \( \frac{d}{dx} \sqrt{x^3 + 1} \)

**Solution:** \( \frac{d}{dx} (x^3 + 1)^{1/2} = \frac{1}{2} (x^3 + 1)^{-1/2} \cdot 3x^2 = \frac{3}{2} x^2 (x^3 + 1)^{-1/2}. \)

(b) \( \frac{d}{dx} \ln(x^3 + 1) \)

**Solution:** \( \frac{d}{dx} \ln(x^3 + 1) = \frac{3x^2}{x^3 + 1}. \)

(c) \( \frac{d}{dx} \frac{e^x}{x} \)

**Solution:** \( \frac{d}{dx} \frac{e^x}{x} = \frac{xe^x - e^x}{x^2}. \)

(d) \( \frac{d}{dx} (x - 3)^3 (x^2 + 1)^4 (x - 8)^{12} \)

**Solution:** Use logarithmic differentiation. First, take ln of both sides to get \( \ln y = \ln (x - 3)^3 (x^2 + 1)^4 (x - 8)^{12} = 3 \ln (x - 3) + 4 \ln (x^2 + 1) + 12 \ln (x - 8). \) Then differentiate both sides to get

\[
\frac{y'}{y} = 3 \frac{1}{x - 3} + 4 \frac{2x}{x^2 + 1} + 12 \frac{1}{x - 8},
\]

so \( y' = y \left( \frac{3}{x - 3} + \frac{8x}{x^2 + 1} + \frac{12}{x - 8} \right). \)
7. (15 points) The quantity demanded each month of the Sicard wristwatch is related to the price by the equation

\[ p = \frac{50}{0.01x^2 + 1} \]

for \( 0 \leq x \leq 20 \) where \( p \) is measured in dollars and \( x \) is measured in units of a thousand.

(a) Find the demand when the price is set at $25 per watch.

**Solution:** Solve the equation \( 25 = \frac{50}{0.01x^2 + 1} \) to get \( x = 10 \).

(b) Recall the revenue function is the product of the price and the number of units sold. Find the revenue function \( R(x) \).

**Solution:** \( R(x) = xp = \frac{50x}{0.01x^2 + 1} \)

(c) Use the results of part b. to find the number of (thousands of) units needed to maximize the revenue.

**Solution:** Note that \( R'(x) = xp = \frac{50(0.01x^2+1) - 0.02x(50x)}{(0.01x^2+1)^2} \), which has only one zero in the interval \([0, 20]\), and that critical point is \( x = 10 \). Note that \( R'(10+) < 0 \) and \( R'(10-) > 0 \) so the first derivative test tells us that \( R \) has a maximum at \( x = 10 \). Note that \( R(10) = 10 \cdot 25 = 250 \), in thousands of dollars.
8. (15 points) Four identical $x \times x$ square corners are cut from a 14 × 18 inch rectangular piece of metal, and the sides are folded upward to build a box.

(a) What is the volume of the box that results when the corners cut are 1 × 1.
   Solution: The volume is the product length × width × height = 12 × 16 × 1 = 192.

(b) Let $V(x)$ denote the volume of the box when the $x \times x$ corners are removed. Find $V(2)$ and $V(3)$.
   Solution: Note that $V(x) = (14 - 2x)(18 - 2x)(x)$ and $V(2) = 10 \cdot 14 \cdot 2 = 280$ and $V(3) = 8 \cdot 12 \cdot 3 = 288$.

(c) What is the implied domain of $V$?
   Solution: The domain of $V$ is [0, 7].

(d) Find $V'(x)$.
   Solution: $V'(x) = \frac{d}{dx}(14 - 2x)(18x - 2x^2) = -2(18x - 2x^2) + (18 - 4x)(14 - 2x) = 4(3x^2 - 32x + 63)$.

(e) Find the critical points of $V(x)$.
   Solution: Use the quadratic formula to get $x = \frac{32 \pm \sqrt{32^2 - 4 \cdot 3 \cdot 63}}{6}$, one of which belongs to the domain. This number, $x = \frac{32 - \sqrt{268}}{6} \approx 2.604$.

(f) What value of $x$ makes the value of $V$ maximum? Estimate within .01 the maximum value of $V$.
   Solution: $V(2.604) \approx 292.864 \approx 292.86$. 
