Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set, $A = \{0, 1, 2, 3, 4, 5\}$, and $B = \{4, 5, 6, 7\}$. Find each of the following numbers.

1. $|A \times B|$
   Solution: $|A \times B| = |A| \times |B| = 6 \times 4 = 24.$

2. $|\overline{A} \times \overline{B}|$
   Solution: $|\overline{A} \times \overline{B}| = |\overline{A}| \times |\overline{B}| = 4 \times 6 = 24.$

3. $|A \times B|$
   Solution: $|A \times B| = |A| \times |B| = 6 \times 6 = 36.$

4. $|\overline{A} \times B|$
   Solution: $|\overline{A} \times B| = |\overline{A}| \times |B| = 4 \times 4 = 16.$

5. $|A \times B| + |\overline{A} \times \overline{B}| + |A \times \overline{B}| + |\overline{A} \times B|$
   Solution: $|A \times B| + |\overline{A} \times \overline{B}| + |A \times \overline{B}| + |\overline{A} \times B| = 24 + 24 + 36 + 16 = 100.$
   Another way to see this is to note that $U \times U$, which has 100 elements, can be decomposed (partitioned) into the four sets $A \times B, \overline{A} \times \overline{B}, A \times \overline{B}$ and $\overline{A} \times B$.

6. $|(A \cap B) \times (A \cup B)|$
   Solution: $|(A \cap B) \times (A \cup B)| = |(A \cap B)| \times |(A \cup B)| = 2 \times 8 = 16.$

7. $|(A \cap B) \times (A \cup \overline{B})|$
   Solution: $|(A \cap B) \times (A \cup \overline{B})| = 2 \times 2 = 4.$

8. $|\{T : T \subseteq U, |T \cap A| = 2 \text{ and } |T \cap B| = 2\}|$. In other words, how many subsets $T$ of $U$ have exactly two elements in common with $A$ and exactly two elements in common with $B$. Hint: start this problem by listing a few of the sets with the desired properties, and then ask yourself ‘how much freedom is there in choosing such sets?’

Solution: We’ll look at three types of sets based on how $T$ intersects $A \setminus B, A \cap B,$ and $B \setminus A$. Case 1. $|T \cap A \cap B| = 2$. There are just four subsets of this type, $\{4, 5\}, \{4, 5, 8\}, \{4, 5, 9\},$ and $\{4, 5, 8, 9\}$. Next suppose $|T \cap (A \setminus B)| = |T \cap (B \setminus A)| = 2$ and $|T \cap (A \cap B)| = 1$. There are 4 ways to choose the member of $A \setminus B$, 2 ways to choose the member of $A \cap B$, 2 ways to choose the member of $B \setminus A$ and 4 ways to attach a subset of $\{8, 9\}$. Finally we could have $|T \cap (A \setminus B)| = 2 = |T \cap (B \setminus A)|$, in which case there are 6 ways to choose the two elements of $A$ and 1 way to choose the 2 elements of $B$, and then 4 ways to finish building the set. The sum of these is $4 + 64 + 24 = 92$ ways to build $T.$