May 9, 2001

Your name ________________________________

It is important that you show your work.

1. Let $\mathcal{U} = \{1,2,\ldots,20\}$ be the universal set. Let $A = \{1,2,\ldots,10\}$, $B = \{11,12,\ldots,20\}$ and $C = \{8,9,10,11,12\}$. Find the following cardinalities
   (a) $|A \times A) \cup (B \times B) \cup (C \times C)|$
   (b) $|(\mathcal{U} \times \mathcal{U}) - (A \times A)|$
   (c) $|(A \cap C) \times (B \cap C)|$
   (d) How many subsets $S$ of $\mathcal{U}$ satisfy both $|S \cap A| = 2$ and $|S \cap B| = 3$?
   (e) How many subsets $S$ of $\mathcal{U}$ satisfy $|S \cap A| = |S \cap B| = |S \cap C| = 2$? For example, $\{2,8,11,15\}$ and $\{9,10,14,15\}$ both satisfy these conditions.
   (f) $|A \times B \times C|$

2. Let $A = \{1,2,3,4\}$
   (a) How many relations on $A$ are there? It may help to think of a relation on $A$ as a 4 by 4 Boolean matrix.
   (b) How many relations on $A$ are reflexive?
   (c) How many relations on $A$ are symmetric?
   (d) How many relations on $A$ are both reflexive and symmetric?
   (e) How many relations on $A$ are reflexive, symmetric and transitive?
   (f) Let $R = \{(1,2),(2,3),(3,4),(2,1)\}$. How many ordered pairs belong to the smallest transitive relation $R_t$ that contains $R$. The relation $R_t$ is called the transitive closure of $R$. The question is, what is $|R_t|$?

3. (a) Show that among any nine points $P_1, P_2, P_3, \ldots, P_9$ in space, where $P_i = (x_i, y_i, z_i)$ and $0 \leq x_i \leq 2$, $0 \leq y_i \leq 2$, $0 \leq z_i \leq 2$, there is some pair whose distance apart is at most 1.75. In other words, if you squeeze 9 points into a $2 \times 2 \times 2$ box, at least two of them must end up fairly close to each other. Recall that the distance between $(0,0,0)$ and $(1,1,1)$ is given by $d = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \approx 1.73$.
   (b) Show that the result is not true if only eight points are selected.

4. Counting Rectangles. Consider the $3 \times 4$ grid of squares shown:
To count the number of rectangle bounded by gridlines, notice that each rectangle is determined by its top and bottom lines together with its left and right bounding lines. For example, \{a, d\}, \{2, 4\} determines the middle 3 by 2 rectangle in the figure. Use this idea to show that there are 60 rectangular regions. Let \( P \) denote the set of rectangular subregions of the grid.

![Grid Diagram](image)

The problem is to count the number of rectangular subregions of grid \( P \) by first counting the rectangles in certain regions of \( P \).

(a) Let \( A \) denote the collection of rectangular regions of the grid

![Region A Diagram](image)

How many rectangular regions are bounded by gridlines in \( A \)?

(b) Let \( B \) denote the set of rectangular subregions of

![Region B Diagram](image)

Find \( |B| \).

(c) Finally, let \( C \) denote the collection of rectangular subregions of the grid

![Region C Diagram](image)
Compute $|C|$.

(d) Use the inclusion-exclusion principle to find $|P|$ in terms of $|A|$, $|B|$, and $|C|$.

5. The number $N = 5^{28} \cdot 7^{12}$ has 30 digits (in decimal notation). Prove that some digit appears at least four times in the representation as follows:

(a) Carefully write out the negation of the conclusion.
(b) If the negation is true what would be the sum of the digits.
(c) What would the truth of the negation imply about the divisibility of $N$ by 9.
(d) Why can $N$ not be a multiple of 9?

6. Use the Euclidean algorithm to solve the decanting problem for decanters of sizes 217 and 475. In other words, find integers $x$ and $y$ such that $\gcd(217, 475) = 217x + 475y$.

7. (Coin flips/probability) A fair coin is flipped seven times. The sample space for this experiment is given by

$S = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mid x_i \in \{H, T\} \text{ for } i = 1, 2, \ldots 7\}$.

Of course, $S$ has $|S| = 2^7 = 128$ elements. Let $A_1 = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mid x_1 = H\}$, $A_2 = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mid x_2 = H\}$, and in general

$A_i = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mid x_i = H\}$,

for $i = 1, 2, \ldots 7$. In other words, $A_i$ is the event that the $i^{th}$ flip results in heads. Let $P(E)$ denote the probability of event $E$.

(a) What is $|A_1|$, and what is $P(A_1)$?
(b) What is $|A_1A_2A_3|$?
(c) What is the probability that exactly three of the seven flips result in heads. Your answer is the number $p_3$ referred to in the next part.
(d) Find the probability $p_i$ of getting exactly $i$ heads in the seven flips, for $i = 0, 1, 2, 3, 4, 5, 6, \text{ and } 7$.

8. John has 1 penny, 3 nickels, 2 dimes, 3 quarters, and 2 dollars. For how many different amounts can John make an exact purchase (with no change required)? Hint notice that some amounts can be made in several ways.
9. (a) Give an example that shows that the union $R \cup T$ of two transitive relations $R$ and $T$ on the set $A$ need not be transitive.

(b) Recall that for any relation $R$ on a set $A$, the relation $R^{-1} = \{(y, x) \mid (x, y) \in R\}$. In other words, to get $R^{-1}$ flip all the pairs of $R$ around. Prove that if a relation $R$ is transitive, then so is $R^{-1}$.

(c) Prove that the compliment $\overline{R}$ of a symmetric relation $R$ is also symmetric.

10. Five cards are selected at random from a deck of 52 ordinary playing cards (there are four suits each with 13 denominations). Let $P(E)$ denote the probability of event $E$.

(a) What is the probability that all five cards selected are red?

(b) What is the probability that all five cards selected are hearts?

(c) What is the probability that there are three hearts and two clubs?

(d) What is the probability that each of the four suits are represented among the five cards?

(e) Let $E_i$ denote the event that exactly $i$ of the five cards are red. Find $P(E_0), P(E_1), P(E_2), P(E_3), P(E_4)$ and $P(E_5)$?
11. Find the base 4 representation of each of the following numbers.

(a) The decimal (ie, base ten) numeral 2002
(b) $2^9 + 2^7 + 2^5 + 2^3 + 1$
(c) $3 \cdot 16^3 + 5 \cdot 16 + 11 \cdot 16^{-2}$
(d) The decimal fraction 0.7.
(e) Explain how you can find the base 2 representation of a base 4 numeral without converting it into a decimal first.