1. (15 points) Solve the decanting problem for containers of sizes 199 and 179; that is find integers $x$ and $y$ satisfying

$$199x + 179y = d$$

where $d$ is the GCD of 199 and 179.

**Solution:** Repeated divisions followed by substitution results in

$$1 = 20 - 19 = 20 - (179 - 8 \cdot 20) = 9 \cdot 20 - 179 = 9(199 - 179) - 179 = 9 \cdot 199 - 10 \cdot 179,$$

so $x = 9$ and $y = -10$.

2. (15 points) Find digits $a$, $b$, and $c$ (between 0 and 4) such that $abc_5 = cba_8$, or prove that there are none.

**Solution:** $abc_5 = 25a + 5b + c$ and $cba_8 = 64c + 8b + a$. If $abc_5 = cba_8$, then $25a + 5b + c = 64c + 8b + a$, or $24a - 3b - 63c = 0$. This simplifies to $8a - b - 21c = 0$. The only solution (easily seen by trial and error) is $a = b = 3$ and $c = 1$. Hence $331_5 = 133_8 = 91$. 
3. (15 points) It is known that the exponential function $b^x, b > 1$ eventually grows larger than any polynomial. In particular, $2^n > n^3$ for all $n$ larger than some threshold value $N$.

(a) Find an integer $N$ such that $2^n > n^3$ for all $n \geq N$.

(b) Use the Principle of Mathematical Induction to prove that your value of $N$ is correct; that is, $2^n > n^3$ for all $n \geq N$. Note that $(n + 1)^3 = n^3 + 3n^2 + 3n + 1$.

Solution: Let $P(n)$ be $2^n > n^3$.

$P(9)$ is false but $P(10)$ states that $2^{10} > 10^3$, which is true. To prove $P(n) \rightarrow P(n + 1)$, let $n$ be at least 10 and assume $P(n)$. Starting with the left side of $P(n + 1)$, we have $2^{n+1} = 2 \cdot 2^n > 2 \cdot n^3 > n^3 + 10n^2 = n^3 + 3n^2 + 7n^2 > n^3 + 3n^2 + 70n \geq n^3 + 3n^2 + 3n + n > n^3 + 3n^2 + 3n + 1 = (n + 1)^3$, the last term of which is the right side of $P(n + 1)$.

4. (15 points) The sequence of Fibonacci numbers $f_0, f_1, f_2, \ldots$ is defined by the rule $f_0 = 0, f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Prove that $f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1$.

Solution: Let $P(n)$ be $f_0 + f_2 + f_4 + f_6 + \cdots + f_{2n} = f_{2n+1} - 1$.

$P(0)$: $P(0)$ states that $f_0 = f_1 - 1$, which is true since $0 = 1 - 1$.

$P(n) \rightarrow P(n + 1)$: Suppose the $P(n)$ is true; i.e., $f_0 + f_2 + f_4 + f_6 + \cdots + f_{2n} = f_{2n+1} - 1$. We must show that $f_0 + f_2 + f_4 + f_6 + \cdots + f_{2(n+1)} = f_{2(n+1)+1} - 1$, i.e., $f_0 + f_2 + f_4 + f_6 + \cdots + f_{2n+2} = f_{2n+3} - 1$:

$$f_0 + f_2 + f_4 + f_6 + \cdots + f_{2n} + f_{2n+2} =$$

$$(f_0 + f_2 + f_4 + f_6 + \cdots + f_{2n}) + f_{2n+2} =$$

$$f_{2n+1} + f_{2n+2} =$$

$$f_{2n+1} + f_{2n+2} - 1 = f_{2n+3} - 1.$$
5. (20 points) Suppose \( S = \{1, 2, \ldots, 20\} \). A subset \( T \) of size 3 is randomly selected.

(a) Find the probability that \( T \) consists of two odd numbers and one even number.

(b) Find the probability that \( T \) consists of three prime numbers.

(c) Find the probability that the three numbers in \( T \) have a sum that is less than 9.

(d) Find the probability that \( T \) has at least one even number in it.

Solution: There are \( C(20, 3) = 1140 \) subsets of size 3.

(a) \( p(T \) contains two odds and 1 even) = \( \frac{C(10, 2) \cdot C(10, 1)}{C(20, 3)} = \frac{450}{1140} \approx 0.394 \)

(b) \( p(T \) consists of three primes) = \( \frac{C(8, 3)}{C(20, 3)} = \frac{56}{1140} \approx 0.0491 \)

(c) \( p(\) sum of the integers in \( T \) is less than 9) = \( \frac{4}{C(20, 3)} \approx 0.0035 \)

(d) \( p(T \) has one even number in it) = \( 1 - p(T \) has no even numbers in it) = \( 1 - \frac{C(10, 3)}{C(20, 3)} \approx 0.8947. \)
6. (35 points) A discrete math class has 10 women and 7 men.

(a) How many 5 element subsets does the class have?

**Solution:** \( \binom{17}{5} = 6188. \)

(b) How many ways are there to choose a committee of size 5 consisting entirely of women?

**Solution:** \( \binom{10}{5} = 252. \)

(c) How many ways are there to choose a committee of size 5 consisting of 4 women and 1 man?

**Solution:** \( \binom{10}{4} \binom{7}{1} = 210 \cdot 7 = 1470. \)

(d) How many ways are there to choose a committee of size 5 consisting of 3 women and 2 men?

**Solution:** The number of ways to choose 3 women is \( C(10, 3) \) and the number of ways to choose 2 men is \( C(7, 2) \). Using the product rule to choose 3 women and 2 men, the answer is \( C(10, 3) \cdot C(7, 2) = 2520. \)

(e) How many ways are there to choose a committee of size 5 consisting of 2 women and 3 men?

**Solution:** \( \binom{10}{2} \binom{7}{3} = 45 \cdot 35 = 1575. \)

(f) How many ways are there to choose a committee of size 5 consisting of 1 woman and 4 men?

**Solution:** \( \binom{10}{1} \binom{7}{4} = 10 \cdot 35 = 350. \)

(g) How many ways are there to choose a committee of size 5 consisting of 5 men?

**Solution:** \( \binom{10}{0} \binom{7}{5} = 1 \cdot 21 = 21. \) Note that the sum of the answers to (b) through (g) is 6188.
7. (20 points) Consider the grid of unit squares below.

(a) How many square regions are bounded the grid lines?

Solution: Count the top right corners of the unit squares, then the $2 \times 2$ squares, etc. to get $49 + 36 + 25 + 16 + 9 + 4 + 1 = 140$.

(b) How many rectangular regions are bounded by grid lines?

Solution: Pick two vertical lines for the left and right boundaries in $\binom{8}{2} = 28$ ways, then pick two horizontal lines to the top and bottom boundaries, also in 28 ways, for a total of $28 \cdot 28 = 784$ rectangular regions.

(c) How many square regions containing the shaded square are bounded the grid lines?

Solution: Again count the upper left corners to get $1 + 4 + 9 + 16 = 30$ squares.

(d) How many rectangular regions containing the shaded square are bounded by grid lines?

Solution: Its just $4^4 = 256$ because you must pick a vertical line among the first 4 and one from among the last four, etc.
8. (35 points) Let $S = \{1, 2, 3, 4, 5, 6, 7\}$.

(a) How many non-empty subsets does $S$ have?
   Solution: $S$ has $2^7 - 1 = 127$ non-empty subsets.

(b) How many subsets of $S$ have no even numbers as members?
   Solution: There are $2^4 = 16$ subsets with no even members.

(c) How many subsets of $S$ have exactly 4 elements?
   Solution: There are $\binom{7}{4} = 35$ four-element subsets of $S$.

(d) How many four-element subsets of $S$ contain exactly two odd numbers?
   Solution: The number of such subsets is $\binom{4}{2} \cdot \binom{3}{2} = 6 \cdot 3 = 18$.

(e) How many four-digit numbers can be made using the digits of $S$ if a digit may be used repeatedly?
   Solution: There are $7^4 = 2401$ such four-digit numbers.

(f) How many four-digit numbers can be made using the digits of $S$ if a digit may be used only once?
   Solution: There are $P_7^4 = 840$ such four-digit numbers.

(g) How many even four-digit numbers bigger than 3000 can be made using the digits of $S$ if a digit may be used only once?
   Solution: There are three types $xxxx2$, $xxxx4$ and $xxxx6$. There are $5 \cdot 5 \cdot 4 \cdot 3 = 300$ of the first type and $4 \cdot 5 \cdot 3 \cdot 2 = 120$ of each of the other two types for a total of $300 + 120 + 120 = 540$. 


9. (10 points) Suppose that if $a, b, c, d$ and $e$ are five different integers.

(a) Must some pair of them differ by a multiple of 4? Explain your answer in detail.

Solution: Consider the remainder when each of the five numbers are divided by 4. There are four possible remainders. By PhP, there must be at least two duplicate remainders, hence two numbers that differ by a multiple of 4.

(b) Must any four different integers include a pair which differ by a multiple of 4?

Solution: No, the set $\{0, 1, 2, 3\}$ is a four-element set no two members of which differ by a multiple of 4.

10. (20 points) Let $R$ be the following relation defined on the set $\{a, b, c, d\}$:

$$R = \{(a, a), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, b), (c, c), (d, b), (d, d)\}.$$ 

Determine whether $R$ is:

(a) reflexive

Solution: $R$ is reflexive since $R$ contains $(a, a), (b, b), (c, c), (d, d)$.

(b) symmetric

Solution: $R$ is not symmetric since $(a, c) \in R$, but $(c, a) \notin R$.

(c) antisymmetric

Solution: $R$ is not antisymmetric since both $(b, c) \in R$ and $(c, b) \in R$ and $b \neq c$.

(d) transitive

Solution: $R$ is not transitive since $(a, c) \in R$ and $(c, b) \in R$, but $(a, b) \notin R$. 


11. (15 points) Suppose \( A, B, \) and \( C \) are sets of integers such that \( B \cap C = \emptyset, |B| = 15, |C| = 12, |A \cap B| = |A \cap C| = 2, \) and \( |A \cup B| = 28 \). Find each of the following:

(a) \( |A \cap \overline{C}| \)

**Solution:** Draw a Venn diagram. You’ll see that \( |A \cap \overline{C}| = 13 \).

(b) \( |(A \cup B) \cap \overline{C}| \)

**Solution:** From the Venn diagram, \( |(A \cup B) \cap \overline{C}| = 26 \).

(c) \( |A \cup B \cup C| \)

**Solution:** From the Venn diagram, \( |A \cup B \cup C| = 38 \).
12. (25 points) Let \( A = \{1, 2, 3\} \).

(a) Give an example of a \( 3 \times 3 \) boolean matrix which represents a relation on \( A \) that is both symmetric and antisymmetric. Which entries of the matrix can be either 0 or 1?

**Solution:** All the off-diagonal entries must be 0. The entries on the diagonal can be either 0 or 1.

(b) Use the information in (a) to count the number of relations on \( A \) which are both symmetric and antisymmetric. (Remember that there are \( 2^9 \) relations on \( A \)).

**Solution:** There are \( 2^3 = 8 \) relations that are both antisymmetric and symmetric.

(c) How many relations on \( A \) are antisymmetric?

**Solution:** There are \( 2^3 \cdot 3^3 = 216 \) relations that are antisymmetric because each pair of symmetrically opposite pairs can be both 0, or one can be 0 and the other 1 (three ways, times 3 such positioned pairs).

(d) How many relations on \( A \) are symmetric?

**Solution:** There are \( 2^6 = 64 \) symmetric relations. Construct the \( 3 \times 3 \) boolean matrix by first deciding on each of the entries on or above the diagonal. Then the entries below the diagonal are uniquely determined.
(e) How many relations on $A$ are reflexive, symmetric, and transitive?

**Solution:** Here we are just counting the number of equivalence relations on a three element set. There are exactly 5 of them, and we can get five by counting the partitions of the set \{1, 2, 3\}.
13. (20 points) Let $A = \{1, 2, 3, 4\}$. Find examples of relations on $A$ which satisfy each of the following collections of conditions:

(a) Reflexive and symmetric and not transitive.

**Solution:** $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (2, 3), (3, 2)\}$ is both reflexive and symmetric but not transitive.

(b) Reflexive and antisymmetric and not transitive.

**Solution:** $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3)\}$ is both reflexive and antisymmetric but not transitive.

(c) Symmetric and transitive and not antisymmetric.

**Solution:** $R = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$ is both reflexive and transitive but not antisymmetric.

(d) Symmetric and not transitive and not reflexive.

**Solution:** $R = \{(1, 2), (2, 1)\}$ is symmetric but not reflexive and not transitive.