1. (10 points) Find the base 6 representation of 2000.
   \textbf{Solution:} \(13132_6\)

2. (10 points) Find the base 6 representation of \(1/5\).
   \textbf{Solution:} \(0.1_6\)

3. (10 points) Find a pair of relatively prime integers \(m\) and \(n\) for which \(m/n = 1.25\).
   Two numbers are relatively prime if their greatest common divisor is 1.
   \textbf{Solution:} \(m = 37\) and \(n = 30\).

4. (10 points) Find a base 7 digit \(d\) such that \(2d16_8 = d405_7\).
   \textbf{Solution:} \(2d16_8 = 1024 + 64d + 8 + 6 = 1038 + 64d\). On the other hand, \(2d16_8 = d405_7 = 343d + 196 + 5 = 343d + 201\). Therefore, \(343d - 64d = 837\), which holds when \(d = 3\).
5. (10 points) Find the best (winning) move in the game of Bouton’s Nim (17, 13, 12, 11).

Solution: take 7 from the pile with 17, leaving the position (10, 13, 12, 11).

6. (12 points) Let $M = 161,161$ and let $N = 12376$.

(a) Compute $LCM(M, N)$

Solution: $LCM = 2^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23$.

(b) Compute $GCD(M, N)$

Solution: $GCD = 7 \cdot 13 = 91$.

(c) Find the number of divisors of $M$.

Solution: $M = 7^2 \cdot 11 \cdot 13 \cdot 23$ and $N = 2^3 \cdot 7 \cdot 13 \cdot 17$ so 1. $LCM = 2^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23$, $GCD = 7 \cdot 13$, and $M$ has $(2 + 1)(1 + 1)(1 + 1)(1 + 1) = 24$ divisors.
7. (20 points) Look at the four equations below.

\[
\begin{align*}
2 &= 2 \cdot 1 \\
2 + 4 &= 3 \cdot 2 \\
2 + 4 + 6 &= 4 \cdot 3 \\
2 + 4 + 6 + 8 &= 5 \cdot 4
\end{align*}
\]

a. Write the next three equations in the sequence.

Solution:

\[
\begin{align*}
2 + 4 + 6 + 8 + 10 &= 6 \cdot 5 \\
2 + 4 + 6 + 8 + 10 + 12 &= 7 \cdot 6 \\
2 + 4 + 6 + 8 + 10 + 12 + 14 &= 8 \cdot 7
\end{align*}
\]

b. If the four equations above correspond to \( k = 1, 2, 3, \) and \( 4, \) what is the \( n \)th equation?

Solution:

\[
2 + 4 + 6 + 8 \ldots + 2n = (n + 1) \cdot n
\]

c. Prove by mathematical induction that the \( n \)th equation is true for all integers \( n \geq 1. \)

Solution: The base case: \( 2 = (2 - 1) \cdot 2. \) Assume \( P(n) : 2 + 4 + 6 + 8 \ldots + 2n = (n + 1) \cdot n. \) To prove \( P(n + 1) : 2 + 4 + 6 + 8 \ldots + 2n + 2(n + 1) = (n + 2) \cdot (n + 1), \) start with the left side and replace the sum of the first \( n \) terms with the right side of \( P(n). \) Thus \( 2 + 4 + 6 + 8 \ldots + 2n + 2(n + 1) = (2 + 4 + 6 + 8 \ldots + 2n) + 2(n + 1) = (n + 1) \cdot n + 2(n + 1) = (n + 1)(n + 2), \) which is the right side of \( P(n + 1). \) By mathematical induction, it follows that \( P(n) \) is true for all \( n \geq 1. \)

8. (10 points) Find the representations of the integers 1 through 13 in base \(-6.\)

Solution: 1, 2, 3, 4, 5, 150, 151, 152, 153, 154, 155, 140, 141.
9. (15 points) Solve the equation $123x + 456y = 3$ for integers $x$ and $y$.

**Solution:** Use the Euclidean Algorithm to find that $x = -63$ and $y = 17$.

10. (13 points) Prove that $2^n \leq n!$ for all integers $n \geq 4$.

**Solution:** First note that $2^4 = 16 < 4! = 24$, so the base case holds. Next assume that $P(n) : 2^n \leq n!$. Then $P(n + 1)$ is the statement $2^{n+1} \leq (n + 1)!$, where $n \geq 4$. But $2^{n+1} = 2 \cdot 2^n \leq 2 \cdot n! < (n + 1) \cdot n! = (n + 1)!$, because $n \geq 4$. 