It is important that you show your work. There are 125 points available on this test.

1. (15 points) Find a pair of integers $m$ and $n$ such that $m/n$ is reduced and $m/n = 21.364$.

2. (20 points)
   
   (a) Find the base 6 representation of 129.
   
   (b) Find the base -6 representation of 129.
   
   (c) Find the base 2 representation of 6.125.

3. (20 points)
   
   (a) Use the division algorithm to find the unique integers $r$ and $q$ satisfying
   
   \[ 377 = 39q + r \quad \text{and} \quad 0 \leq r < 39. \]

   (b) Solve the decanting problem for containers of sizes 377 and 39; that is find integers $x$ and $y$ satisfying
   
   \[ 377x + 39y = d \]

   where $d$ is the GCD of 39 and 377. containers of sizes 387 and 39; that is
4. (20 points) Notice that

\begin{align*}
1 &= 1 = 1^2 \\
1 + 3 &= 4 = 2^2 \\
1 + 3 + 5 &= 9 = 3^2 \\
1 + 3 + 5 + 7 &= 16 = 4^2
\end{align*}

(a) List the next three equations suggested by the pattern.

(b) Given that the four equations above are the 1st, 2nd, 3rd, and 4th, write the \( n \)th equation of the sequence. Notice that in the 4th equation, the last summand is 7 (not 4).

(c) Use mathematical induction to prove that the \( n \)th equation is true for all positive integer values of \( n \).
5. (15 points) Divisors Let $p$, $q$, and $r$ be three different prime numbers. In terms of $p$, $q$, and $r$, compute

(a) $\text{GCD}(p^3q^2r, p^2qr^3)$
(b) $\text{LCM}(p^3q^2r, p^2qr^3)$
(c) the number of divisors of $p^3q^2r$.

6. (20 points) State the Fundamental Theorem of Arithmetic. Then use it to give an argument that the square root of 2 is irrational. Why is it not possible to prove that $\sqrt{4}$ is not rational using this method? Elaborate.

7. (15 points) Prove that for any integer $n \geq 5$, $2^n > n^2$. 