1. (15 points) Solve the decanting problem for containers of sizes 180 and 266; that is, find integers \(x\) and \(y\) satisfying

\[180x + 266y = d\]

where \(d\) is the GCD of 180 and 266.

**Solution:** Repeated division yields

\[2 = 8 - 6 = 8 - (86 - 10 \cdot 8) = 8 - 86 + 10 \cdot 8 =
\]

\[11 \cdot 8 - 86 = 11(180 - 2 \cdot 86) - 86 = 11 \cdot 180 - 22 \cdot 86 - 86 = 11 \cdot 180 - 23(266 - 180) =
\]

\[34 \cdot 180 - 23 \cdot 266.\]

2. (12 points) Let \(N = 17017\).

(a) What is the prime factorization of \(N\)?

**Solution:** \(17017 = 7 \cdot 11 \cdot 13 \cdot 17.\)

(b) What is the remainder when \(N\) is divided by 9?

**Solution:** The sum of the digits is 16, so \(N \equiv 16 \equiv 7 (\text{mod } 9).\)

(c) What is the remainder when \(N\) is divided by 11?

**Solution:** Since 11 is a factor of \(N\), the remainder is 0.

(d) How many (positive integer) divisors does \(N\) have?

**Solution:** Since \(N\) factors as a product of four distinct primes, it has

\[2 \cdot 2 \cdot 2 \cdot 2 = 16\] factors.
3. (10 points) Find a pair of relatively prime integers \(m\) and \(n\) such that \(\frac{m}{n} = 0.857142\).

\[\text{Solution:}\] Call the number \(x\). Then \(10^6x - x = 999999x = 857142\), so \(x = 857142/999999\). This fraction reduces to \(6/7\).

4. (10 points) Find the base 5 representation of the number 717 in two ways.

\[\text{Solution:}\] By repeated subtraction, \(717 = 625 + 92 = 625 + 3 \cdot 25 + 17 = 5^4 + 3 \cdot 5^2 + 3 \cdot 5 + 2 = 10332_5\). Also, by repeated division, we get the remainders 2, 3, 3, 0, and 1.

5. (10 points) Find the base 5 representation of the number \(2/3\).

\[\text{Solution:}\] Repeated multiplication yields: \(2/3 \cdot 5 = 10/3 = 3 + 1/3\). Then \(1/3 \cdot 5 = 5/3 = 1 + 2/3\), and the repetition becomes clear. Thus \(2/3 = 0.3131_5\).
6. (20 points) Notice that

\begin{align*}
1 &= 1 \\
1 + 3 &= 4 \\
1 + 3 + 5 &= 9 \\
1 + 3 + 5 + 7 &= 16
\end{align*}

(a) List the next three equations suggested by the pattern.

Solution:

\begin{align*}
1 + 3 + 5 + 7 + 9 &= 25, \\
1 + 3 + 5 + 7 + 9 + 11 &= 36, \\
1 + 3 + 5 + 7 + 9 + 11 + 13 &= 49.
\end{align*}

(b) Given that the four equations above are the 1\textsuperscript{st}, 2\textsuperscript{nd}, 3\textsuperscript{rd}, and 4\textsuperscript{th}, write the \textit{n}\textsuperscript{th} equation of the sequence.

Solution:

\[1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) = n^2.\]

(c) Use mathematical induction to prove that the \textit{n}\textsuperscript{th} equation is true for all positive integer values of \(n\).

Solution: The base case for the proof is the first equation above, \(1 = 1^2\). Given that \(1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) = n^2\) holds for some \(n \geq 1\), consider the left side of \(P(n+1)\). Note that \(1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) + (2(n+1) - 1) = n^2 + (2(n + 1) - 1) = n^2 + 2n + 1 = (n + 1)^2\). By PMI, the proposition \(P(n)\) is true for all \(n \geq 1\).
7. (10 points) You are playing a game of Bouton’s Nim, and the position is 
\((17, 15, 13, 9, 6)\). Is this a safe position? If not, find the winning move. 

**Solution:** The binary configuration is 
\[
\begin{align*}
6 &= 110 \\
9 &= 1001 \\
13 &= 1101 \\
15 &= 1111 \\
17 &= 10001
\end{align*}
\]
so the three leftmost columns must be balanced. The only winning move is to 
take 4 counters from the pile of 17.

8. (8 points) What is the remainder when \(3^{2002}\) is divided by 5? 

**Solution:** Note that \(3^2 \equiv -1 \pmod{5}\), so \(3^{2002} = (3^2)^{1001} \equiv (-1)^{1001} \equiv -1 \equiv 4 \pmod{5}\). So the remainder is 4.

9. (10 points) Find digits \(a\) and \(b\) for which the number 11\(ab\)3 is a multiple of 99. 

**Solution:** The number must be both a multiple of 9 and of 11. Applying the 
divisibility tests for 9 and 11 gets A. either \(5 + a + b = 9\) or \(5 + a + b = 18\) and 
B. either \(3 + a - b = 0\) or \(3 + a - b = 11\). Only two of these are compatible 
and they lead to \(a = 5\) and \(b = 8\). Thus \(11583 = 99 \cdot 117\) is a multiple of 99.

10. (15 points) 

(a) Suppose \(p, q,\) and \(r\) are different prime numbers. How many (positive 
 integer) divisors does \(N = p^2q^3r^5\) have? 

**Solution:** The number of divisors is given by \(D = (2+1)(3+1)(5+1) = 72\). 

(b) Find the greatest common divisor (GCD) of \(N\) and \(M = pq^4\). 

**Solution:** \(GCD = p^1q^3r^0 = pq^3\). 

(c) Find the least common multiple (LCM) of \(N\) and \(M = pq^4\). 

**Solution:** \(LCM = p^2q^4r^5\).

11. (10 points) Let \(x = \log_2 5\). This means that \(2^x = 5\). If \(x\) is rational, then there 
are integers \(m\) and \(n\) such that \(2^{m/n} = 5\), or equivalently, \(2^m = 5^n\). Explain in 
detail why \(\log_2 5\) is irrational. 

**Solution:** Because \(2^n\) is even for any positive integer \(m\), and \(5^n\) is an odd 
number for any nonnegative integer \(n\), \(2^m \neq 5^n\) for any \(m, n\).