1. (20 points) Let \( P \) denote the compound proposition defined by \( P : (p \rightarrow q) \rightarrow r \) and let \( Q : p \rightarrow (q \rightarrow r) \). Test the associativity of \( \rightarrow \) by determining whether \( P \) and \( Q \) are logically equivalent.

**Solution:**

<table>
<thead>
<tr>
<th></th>
<th>( p )</th>
<th>( q )</th>
<th>( q \rightarrow r )</th>
<th>( p \rightarrow (q \rightarrow r) )</th>
<th>( p \rightarrow q )</th>
<th>( (p \rightarrow q) \rightarrow r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

The two propositions \( P \) and \( Q \) do not agree in the 6th and 8th rows. Thus \( P \) and \( Q \) are not locally equivalent because \( P \Leftrightarrow Q \) is not a tautology.
2. (24 points) Determine the truth value of the following statements if the uni-
verse of discourse of each variable is the set of real numbers.

_1. \( \forall x \exists y (x = y^2) \)
   
   **Solution:** False. Try \( x = -1 \).

_2. \( \forall x \exists y ((x + y = 2) \land (2x - y = 1)) \)
   
   **Solution:** False. Try \( x = 0 \).

_3. \( \exists x (x^2 = -1) \)
   
   **Solution:** False. No real number has a negative square.

_4. \( \forall x \neq 0 \exists y (xy = 1) \)
   
   **Solution:** True. Try \( y = 1/x \).

_5. \( \exists x \exists y (x + y \neq y + x) \)
   
   **Solution:** False. Addition is commutative.

_6. \( \forall x \exists y (x + y = 1) \)
   
   **Solution:** True. Try \( y = x - 1 \).

_7. \( \exists x (x^2 = 2) \)
   
   **Solution:** True. Try \( x = \sqrt{2} \).

_8. \( \exists x \forall y (xy \neq 1) \)
   
   **Solution:** False. There is no such \( x \).

_9. \( \forall x \forall y \exists z (z = \frac{x + y}{2}) \)
   
   **Solution:** True.

_10. \( \exists x \exists y ((x + 2y = 2) \land (2x + 4y = 5)) \)
    
   **Solution:** False. If \( x + 2y = 2 \) then \( 2x + 4y = 4 \).

_11. \( \exists x \forall y (xy = 0) \)
    
   **Solution:** True. Try \( x = 0 \).

_12. \( \forall x \exists y (x^2 = y) \)
    
   **Solution:** True. Try \( y = x^2 \).
3. (20 points) Let $D$ denote the set of all real numbers and let $P$ and $Q$ denote the two-place predicates on $D$ defined by $P(x, y) : x \leq y$ and $Q(x, y) : y \leq x$. Find the truth value of each of the compound propositions.

(a) $\forall x \forall y (P(x, y) \land Q(x, y)) \rightarrow x = y$.

**Solution:** We can translate $P(x, y)$ as $x \leq y$ and $Q(x, y)$ as $y \leq x$. Therefore we have $(x \leq y) \land (y \leq x) \rightarrow x = y$ which is an axiom for the real numbers. This property of $\leq$ is called *antisymmetry*.

(b) $\forall x \forall y (P(x, y) \rightarrow \neg Q(x, y))$.

**Solution:** We have $x \leq y \rightarrow x < y$ which is false for $x = y = 1$.

(c) $\forall x \exists y (P(x, y) \rightarrow Q(x, y))$.

**Solution:** We have $x \leq y \rightarrow y \leq x$ which is true. Take $y = x - 1$.

(d) $\forall x \forall y (P(x, y) \lor Q(x, y)) \rightarrow x \neq y$.

**Solution:** This translates into $(x \leq y) \lor (y \leq x) \rightarrow x \neq y$ which is false when $x = y = 1$.

(e) $\forall x \forall y x \neq y \leftrightarrow (\neg P(x, y) \lor \neg Q(x, y))$.

**Solution:** This translates into $x \neq y \leftrightarrow \neg (x \leq y) \lor \neg (y \leq x)$ which is equivalent to $(x \neq y) \leftrightarrow (y < x \lor x < y)$ which is an axiom for the real numbers (same as item a).
4. (20 points) Notice that

\[
\begin{align*}
\frac{1}{1 \cdot 2} &= \frac{1}{2} \quad & (1) \\
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} &= \frac{2}{3} \quad & (2) \\
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} &= \frac{3}{4} \quad & (3)
\end{align*}
\]

(a) List the next two equations suggested by the pattern.

Solution:

\[
\begin{align*}
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} &= \frac{4}{5} \\
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} &= \frac{5}{6}
\end{align*}
\]

(b) Given that the three equations above are the 1\textsuperscript{st}, 2\textsuperscript{nd}, and 3\textsuperscript{rd}, write the \(n\textsuperscript{th}\) equation of the sequence.

\[
P(n) : \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.
\]
(c) Use mathematical induction to prove that the $n^{th}$ equation is true for all positive integer values of $n$.

**Solution:** Clearly $P(1)$ is true. Assume $P(n)$. To prove $P(n + 1)$:

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2},
\]

start with the left side of $P(n + 1)$.

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n+1)(n+2)} = \\
\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \\
\frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} = \\
\frac{n^2 + 2n + 1}{(n+1)(n+2)} = \frac{n + 1}{n + 2}.
\]

By the Principle of Mathematical Induction, $P(n)$ holds for all $n \geq 1$. 
5. (20 points) Prove that $4^n - 1$ is divisible by 3 for all $n \geq 1$.

Solution: Let $P(n)$ be the predicate defined by $P(n) : 3|4^n - 1$. Now $P(1)$ is $3|4 - 1$ which is true. Assume $P(n)$. In other words, assume that for any $n \geq 1$, $4^n - 1 = 3k$ for some integer $k$. Then $4^{n+1} - 1 = 4 \cdot 4^n - 4 + 4 - 1 = 4(4^n - 1) + (4 - 1) = 4 \cdot 3k + 3 = 3(4k + 1)$ which is a multiple of 3. By the Principle of Mathematical Induction, $P(n)$ holds for all $n \geq 1$.

6. (10 points) In a group of 100 students, the following facts are known:

- 50 take accounting,
- 40 take biology,
- 35 take chemistry,
- 12 take both accounting and biology,
- 10 take accounting and chemistry,
- 11 take chemistry and biology, and
- 5 take all three subjects.

How many take none of the three subjects?

Solution: Use a venn diagram. Alternatively, use the inclusion-exclusion problem. Let $A$, $B$, and $C$ denote the sets of students who study accounting, biology, and chemistry respectively. Then $|A \cup B \cup C| = |A| + |B| + |C| - |AB| - |AC| - |BC| + |ABC| = (50 + 40 + 35) - (12 + 10 + 11) + 5 = 97$, which leaves 3 students studying none of the three subjects.
7. (20 points) Let $Z$ denote the set of all integers. Classify each of the following functions from $Z$ to $Z$ as one-to-one and onto, one-to-one and not onto, onto and not one-to-one, neither onto nor one-to-one. Prove your answers.

(a) Let $f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n - 1 & \text{if } n < 0 \end{cases}$

**Solution:** $f$ is one-to-one but not onto. The number $-1$ is not in the range of $f$. If $0 \leq n_1 < n_2$, then $f(n_2) > f(n_1)$. If $n_2 < n_1 < 0$, then $f(n_2) > f(n_1)$. If $n_1 \geq 0 > n_2$, then $f(n_1)$ is even and $f(n_2)$ is odd, so $f(n_1) \neq f(n_2)$. Thus, if $n_1 \neq n_2$, then $f(n_1) \neq f(n_2)$.

(b) Let $f(n) = \begin{cases} n - 1 & \text{if } n \geq 1 \\ n + 1 & \text{if } n \leq 0 \end{cases}$

**Solution:** $f$ is onto but not one-to-one. Its not one-to-one because $f(1) = f(-1) = 0$. On the other hand, if $m \geq 0$, then $f(m + 1) = m + 1 - 1 = m$ and if $m < 0$, the $f(m - 1) = m - 1 + 1 = m$, so $f$ is onto.

(c) $f(n) = -n$

**Solution:** $f$ is both one-to-one and onto. If $n_1 \neq n_2$, then $f(n + 1) = -n_1 \neq -n_2 = f(n_2)$. So $f$ is one-to-one. If $m \in Z$, $f(-m) = m$, so $f$ is onto.

(d) $f(n) = |n|$

**Solution:** $f$ is neither one-to-one nor onto, since $f(-1) = f(1)$ and $-1$ is not in the range of $f$. 

7