November 6, 2000

This test has 117 points available.

1. (20 points) Let $A$ and $B$ be sets with characteristic functions $f_A$ and $f_B$. Compute the following characteristic functions of the terms of $f_A$ and $f_B$.

(a) $f_{A \cap B}$

Solution: $f_A \cdot f_B$

(b) $f_{A \cup B}$

Solution: $f_A + f_B - f_A f_B$

(c) $f_{\overline{A}}$

Solution: $1 - f_A$

(d) Prove the DeMorgan property $\overline{A \cap B} = \overline{A} \cup \overline{B}$ using the characteristic functions above.

Solution: $f_{\overline{A \cap B}} = 1 - f_{A \cap B} = 1 - f_A f_B$, whereas $f_{\overline{A} \cup \overline{B}} = f_{\overline{A}} + f_{\overline{B}} - f_{\overline{A}} f_{\overline{B}} = (1 - f_A) + (1 - f_B) - (1 - f_A)(1 - f_B) = 2 - f_A - f_B - 1 + f_A + f_B - f_A f_B = 1 - f_A f_B$.

2. (12 points) Among 32 students in a room, 8 study mathematics, 11 study science, and 10 study computer programming. Also, 4 study mathematics and science, 5 study mathematics and computer programming, and 7 study science and computer programming. We know that 2 students study all three subjects. How many of these students study none of the three subjects?

Solution: Let $M$, $S$, and $C$ denote the sets of students who study math, science, and computing respectively and let $U$ be the entire set of 32 students. Then $|M| = 8$, $|S| = 11$, and $|C| = 10$. Also, we have $|MS| = 4$, $|MC| = 5$, and $|SC| = 7$, where $|X|$ denotes the number of elements of the set $X$ and juxtaposition of sets means intersection. Finally, $|MCS| = 2$. Then

$|U| - (|M| + |S| + |C| - |MS| - |MC| - |SC| + |MCS|) = |MCS|^c = 32 - (29 - 16 + 2) = 17$.

3. (15 points) Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7\}$, and $C = \{1, 5, 8\}$. Recall that $\times$ denotes Cartesian product and $\overline{X}$ denotes the complement of $X$ with respect to $U$. Find each of the following. Recall that $A \oplus B = A \setminus B \cup B \setminus A$ denotes the symmetric difference of $A$ and $B$ and that $|X|$ denotes the number of elements of the finite set $X$. 


(a) \(|A \cup B \cup C|

\textbf{Solution:} \(|A \cup B \cup C| = |\{1, 2, 3, 4, 5, 6, 7, 8\}| = |\{0, 9\}| = 2\)

(b) \(|(A \times A) \cap (B \times B) \cap (C \times C)|

\textbf{Solution:} Since \(A \cap B \cap C = \phi\), it follows that \((A \times A) \cap (B \times B) \cap (C \times C)\) is also empty. Thus \(|(A \times A) \cap (B \times B) \cap (C \times C)| = 0\)

(e) \(|(A \oplus B) \cup (B \oplus C) \cup (C \oplus A)|

\textbf{Solution:} Notice that each of the numbers 1, 2, 3, 4, 5, 6, 7, 8 belong to one or two of the sets \(A, B\) and \(C\), so \(|(A \oplus B) \cup (B \oplus C) \cup (C \oplus A)| = 8\).

4. (15 points) Recall that if \(a \equiv b(\text{mod } 9)\) and \(c \equiv d(\text{mod } 9)\), then both \(a + c \equiv b + d(\text{mod } 9)\) and \(a \cdot c \equiv b \cdot d(\text{mod } 9)\).

(a) Prove that each of the numbers \(10^0, 10, 10^2, \ldots, 10^n\) is congruent to 1 modulo 9. IE, \(10^n \equiv 1(\text{mod } 9)\) for all positive integers \(n\).

\textbf{Solution:} Note that \(10^1 = 10 \equiv 1(\text{mod } 9)\). Suppose \(10^{n-1} \equiv 1(\text{mod } 9)\) for some \(n\). Then \(10^n = 10^{n-1} \cdot 10 \equiv 1 \cdot 1 = 1(\text{mod } 9)\), so the proposition is true by mathematical induction.

(b) Use the given properties and part (a) to prove that \(2345678 \equiv 35(\text{mod } 9)\).

In other words, \(2345678\) is congruent to the sum of its digits, modulo 9.

\textbf{Solution:} Note that \(2345678 = 2 \cdot 10^6 + 3 \cdot 10^5 + \cdots + 8 \equiv 2 \cdot 1 + 3 \cdot 1 + \cdots + 8 \cdot 1\) by the property above. By the additive property in the hypothesis, the rest follows.
5. (15 points) Do exactly one of the next two problems.

1. Reproduce the proof that \([0, 1] \approx [0, 1] \times [0, 1]\).
2. Prove that there is no function from \(N = \{1, 2, 3, \ldots\}\) ONTO \([0, 1]\). That is, there is no function from the natural numbers onto the closed unit interval.

**Solution:** 1. Let \(x = 0.x_1x_2\ldots\) and define \(f : [0, 1] \rightarrow [0, 1] \times [0, 1]\) by

\[
f(x) = (0.x_1x_3x_5\ldots, 0.x_2x_4x_6\ldots).
\]

We must prove that \(f\) is both one-to-one and onto. To see that \(f\) is one-to-one, take two different numbers \(x\) and \(y\) in \([0, 1]\), and find the first decimal place where they differ. If \(x_i = y_i\) for \(i = 1, \ldots, k - 1\) and \(x_k \neq y_k\), the \(f(x) \neq f(y)\) because they have different first coordinates if \(k\) is odd and different second coordinates if \(k\) is even. To see that \(f\) is onto, let \((u, v) \in [0, 1] \times [0, 1]\). Suppose \(u = 0.u_1u_2u_3\ldots\) and \(v = 0.v_1v_2v_3\ldots\). Then let \(w = 0.u_1v_1u_2v_2\ldots\). Notice that \(f(w) = (u, v)\). Thus \(f\) is both one-to-one and onto, hence \(f\) is a bijection.

**Solution:** 2. This is Cantor’s diagonalization method. Suppose \(f : N \rightarrow [0, 1]\). Write the binary representation of \(f(1), f(2), \ldots\,\) in a matrix as shown:

\[
\begin{align*}
f(1) &= x_{1,1}x_{1,2}x_{1,3}\ldots \\
f(2) &= x_{2,1}x_{2,2}x_{2,3}\ldots \\
f(3) &= x_{3,1}x_{3,2}x_{3,3}\ldots \\
&\vdots
\end{align*}
\]

Next construct a number \(y\) in \([0, 1]\) that is different from any of the \(f(n)\) as follows; \(y = y_1y_2y_3\ldots\) where \(y_i = 1 - x_{i,i}\). Comparing each \(f(n)\) with \(y\) we can see that they are different in the \(n^{th}\) position, so they are different. Thus \(y\) is not in the image of \(f\).

6. (15 points) Given a set \(C\) of real numbers that satisfies

(a) \(5 \in C \land 8 \in C\) (Both 5 and 8 belong to \(C\)).
(b) \(\forall x \forall y, \ (x \in C \land y \in C) \rightarrow (x - y \in C)\). This means that for any elements \(x\) and \(y\) of \(C\), the difference \(x - y\) also belongs to \(C\).

i Prove that \(11 \in C\).

**Solution:** By (b) and the fact that 5 and 8 are in \(C\), \(8 - 5 = 3\) also belongs to \(C\). By (b) \(8 - 8 = 0 \in C\). Again by (b), \(0 - 3 = -3 \in C\). Finally, by (b), \(8 - (-3) = 11 \in C\).
ii Prove that $C$ contains all the positive integers.

**Solution:** By part (a), $5 \in C$ and $3 \in C$. Therefore, by (b), $5 - 3 = 2 \in C$ and again by (b), $3 - 2 = 1 \in C$. Since $0 \in C$, it follows that $0 - 1 = -1 \in C$. Proceeding by mathematical induction, suppose $n - 1 \in C$. Then $n - 1 - (-1) = n \in C$ again by property (b). Thus $C$ includes all the positive integers, by mathematical induction.

In fact, $C$ contains the set of all integers. To see this note that $8 - 5 = 3$ and $5 - 3 = 2$ both belong to $C$. Then $3 - 2 = 1$ and $2 - 3 = -1$ both belong using the set building property repeatedly. Of course, $8 - 8 = 0$ also belongs, and so does $0 - (-1) = 1$. This leads to $x - (-1) = x + 1 \in C$ whenever $x \in C$. It follows that $C$ contains the set of positive integers.

7. (25 points) Define a sequence $a_1, a_2, \ldots$ of numbers recursively by $a_1 = 1$, $a_2 = 1$, and $a_{n+2} = 3a_{n+1} - 2a_n$, for $n \geq 1$.

(a) Compute from scratch the first 6 terms of the sequence.

**Solution:** The first 6 values are 1.

(b) Find and solve the characteristic equation of the recursion.

**Solution:** Since $a_{n+2} - 3a_{n+1} + 2a_n = 0$, we replace $a_n$ with $r^n$ to get $r^{n+2} - 3r^{n+1} + 2r^n = 0$ and factoring out $r^n$, and then factoring the resulting quadratic yields the characteristic equation $(r - 2)(r - 1) = 0$. Thus $a_n = 1^n$ and $a_n = 2^n$ are both solutions of the recurrence relation.

(c) Find a general solution with (undetermined constants).

**Solution:** The general solution is a linear combination of the two solutions above, $a_n = c_12^n + c_21^n$.

(d) Use the initial conditions to find a specific solution.

**Solution:** Using the initial conditions $a_1 = 1 = a_2$ to solve for $c_1$ and $c_2$ give the unique solution $a_n = 1$ to the recurrence problem.