1. (20 points) Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. For each property below find the number of subsets $T$ of $S$ that have the property.

(a) $T$ has exactly three elements.

(b) $|T| = 5$ and $T$ has three odd members and two even members.

(c) $T$ has no prime number members. Recall that 1 is not prime.

(d) $T$ has at least two odd and at least three even members.
2. (20 points) Again let \( S = \{1, 2, 3, 4, 5, 6, 7, 8\} \). For each property below find the number of five digit numbers that can be constructed using the members of \( S \) as digits and that have the property.

(a) The number has a value of at least 20000.

(b) The number has a value of at least 20000 and the digits are all different.

(c) The number is a multiple of nine and the digits are all different.

(d) The digits of the number are in increasing order from left to right. For example, 13457.
3. (20 points) Let \( Z \) denote the set of all integers, \( Z = \{0, \pm 1, \pm 2, \ldots, \} \), and let \( Z^+ \) denote the set of positive integers.

(a) Find a one-to-one function \( g \) from \( Z^+ \) onto \( Z \).

(b) Prove that your function is one-to-one.

(c) Prove that your function is onto.
4. (20 points) Let $A$ and $B$ be sets with characteristic functions $f_A$ and $f_B$. Compute the following characteristic functions of the terms of $f_A$ and $f_B$.

(a) $f_{A \cap B}$

(b) $f_{A \cup B}$

(c) $f_{\overline{A}}$

(d) Use the properties (a),(b), and (c) to compute the characteristic function of $A \cup (B \cup C)$. How does this anticipate the inclusion-exclusion principle for three sets?

5. (10 points) Suppose $f(n) = 5f\left(\frac{n}{2}\right) + n + 1$ and $f(4) = 40$. Find $f(1)$. 
6. (15 points) Suppose one hundred students are polled about the academic preferences. Let $A$, $B$, and $C$ denote the student who enjoy studying anthropology, biology, and chemistry respectively. For convenience, write $AB$ to mean $A \cap B$, etc. Recall that $|X|$ denotes the number of elements of the set $X$. You are given the following information about the hundred students: $|ABC| = 2, |AB| = 7, |A \cup C| = 39, |ABC^c| = 6, |A| = 25, |B| = 30,$ and $|C| = 20$. How many students do not enjoy any of the three types of courses?
7. (10 points) The points shown are the vertices of a regular hexagon with side length 1 together with the center of the hexagon. How many circles of radius 1 in the same plane have at least two vertices in the set?

8. (15 points) Find the number of integers from 1 to 600 inclusive that are not divisible by any of the numbers 3, 5, and 7
9. (10 points) Solve: $a_n = 3a_{n-1} + 10a_{n-2}$ for $n \geq 2$ with the initial values $a_0 = 0$, $a_1 = 1$. Use your solution to find the value of $a_{10}$. 
10. (10 points) Suppose $U$ is a 14 element set with subsets $A$, $B$, and $C$ satisfying

$|A| = |B| = |C| = 7$, $|AB| = 4$, $|BC| = 3$, $|AC| = 2$, and $|ABC| = 1$. Compute the following cardinalities.

(a) $|A \cup (B \cap C)|$

(b) $|(A \times A) \cup (B \times B) \cup (C \times C)|$

(c) $|(A \cap B) \times (B \cap C) \times (A \cap C)|$

(d) $|(A \cup B \cup C) \times (A \cup B \cup C)|$

(e) $|(A \cup B \cup C) \times (A \cup B \cup C)|$