Your name

There are 142 points available on this test. You must show all your work.

1. (12 points) Solve the decanting problem for containers of sizes 299 and 279; that is find integers $x$ and $y$ satisfying

$$299x + 279y = d$$

where $d$ is the GCD of 299 and 279.

**Solution:** Repeated divisions followed by substitution results in $1 = 20 - 19 = 20 - (279 - 13 \cdot 20) = 14 \cdot 20 - 279 = 14(299 - 279) - 279 = 14 \cdot 299 - 15 \cdot 279$, so $x = 14$ and $y = -15$.

2. (12 points) Find the base 5 representation of each of the numbers below.

(a) 2005

**Solution:** By repeated division, $2005 = 31010_5$

(b) 13.128

**Solution:** By a combination of repeated multiplication and repeated division, $13.128 = 23.031_5$

(c) $1/3$

**Solution:** By repeated multiplication, $1/3 = 0.13_5$

3. (10 points) Find the base $-5$ representation of each of the numbers below.

(a) 2005

**Solution:** By repeated division, $2005 = 44140_{-5}$

(b) 3.128

**Solution:** By trial and error, $3.128 = 3.044_{-5}$
4. (20 points) Let $\mathbb{R}$ denote the real numbers and let $R$ be the relation on $\mathbb{R}$ defined by $xRy \iff \lfloor x \rfloor - \lfloor y \rfloor$ is a multiple of 3. For example $\pi R 0.3$ because $\lfloor \pi \rfloor - \lfloor 0.3 \rfloor = 3$ which is a multiple of 3.

(a) Prove that $R$ is reflexive.
   Solution: Since $\lfloor x \rfloor - \lfloor x \rfloor = 0$, which is a multiple of 3, $R$ is reflexive.

(b) Prove that $R$ is symmetric.
   Solution: $\forall x, y \ (\lfloor x \rfloor - \lfloor y \rfloor) = -(\lfloor y \rfloor - \lfloor x \rfloor)$, it follows that neither or both are multiples of 3.

(c) Prove that $R$ is transitive.
   Solution: Just add the two fractions $(\lfloor x \rfloor - \lfloor y \rfloor)/3$ and $(\lfloor y \rfloor - \lfloor z \rfloor)/3$ to get a whole number.

(d) Since $R$ is an equivalence relation, $\mathbb{R}$ is partitioned into cells by $R$. Describe $[\pi]$.
   Solution: $[\pi]$ is the set of all numbers whose floor is a multiple of 3.

5. (15 points)

(a) Solve the equation $1x \cdot x6 = 1x010_5$ for the missing digit.
   Solution: Interpret to get $9^3 + x9^2 + x \cdot 9 + 6 = 5^4 + x5^3 + 5$ which reduces to $35x = 105$. Therefore $x = 3$.

(b) Find positive integers $b$ and $c$ such that $31000b = 13310c$.
   Solution: Interpret to get $3b^4 + b^3 = c^5 + 3c^4 + 3c^3 + c^2$. Factor both sides to get $b^3(3b + 1) = c^5 + 3c^4 + 3c^3 + c^2 = c^2(c^3 + 3c^2 + 3c + 1)$, so it makes sense to try $c + 1 = b$ in which case $3b + 1 = (b - 1)^2$ and $b = 5$. Therefore $c = 4$. 
6. (15 points) Let \( X = \{1, 2, 3, 4, 5\} \), \( Y = \{6, 7, 8, 9\} \), and \( Z = \{10, 11, 12, 13, 14\} \). Let \( R \) be the relation from \( X \) to \( Y \) defined by \( xRy \iff y = x + 2 \lor y = x + 4 \), and \( S \) be the relation from \( Y \) to \( Z \) defined by \( ySz \iff z = y + 3 \lor z = y + 5 \).

(a) Using the natural order on each of the sets, construct the matrices \( M_R \) and \( M_S \).

\[
M_R = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
\end{pmatrix}
\quad \text{and} \quad
M_S = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
\end{pmatrix}
\]

(b) Compute the product matrix \( M_R \cdot M_S \).

\[
\text{Solution: } M_R \cdot M_S = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 \\
1 & 0 & 2 & 0 & 1 \\
\end{pmatrix}
\]

(c) Find the matrix \( M_{S \circ R} \) of the composition of \( R \) and \( S \). Recall that the notation \( S \circ R \) means that \( R \) is applied first, just as would be the case if \( R \) and \( S \) were functions.

\[
\text{Solution: } M_{S \circ R} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
\end{pmatrix}
\]

7. (10 points) Let \( R \) and \( S \) be transitive relations on a set \( X \). Prove that \( R \cap S \) is a transitive relation.

\[
\text{Solution: } \text{Suppose } xRy \land yRz \land xRz. \ \text{Then } xRy, xSy, yRz \text{ and } ySz \text{ because of the meaning of intersection. By transitivity of } R \text{ and } S \text{ it follows that } xRz \text{ and } xSz. \ \text{Therefore } xR \cap Sz, \text{ which is what we needed to prove.}
\]
8. (12 points) Let $X = \{1, 2, 3\}$. Each relation on $X$ is a $3 \times 3$ Boolean matrix (we assume the natural ordering on $X$).

(a) How many relations are there on $X$?
Solution: $2^9 = 512$.

(b) How many relations are both symmetric and antisymmetric?
Solution: Such a relation can have only loops, and there are $2^3 = 8$ such relations.

(c) How many relations are both antisymmetric and reflexive?
Solution: This type relation must have all of its loops. The only decisions are what to put in the off diagonal locations of the matrix. Thus $(a_{ij}, a_{ji})$ must be $(0, 0), (0, 1), \text{ or } (1, 0),$ three choices. Since there are three pairs of symmetric positions, there are $27$ choices.

(d) How many equivalence relations are there on $X$?
Solution: Five. Count the number of partitions of the set $X$. There’s $123, 1|2|3, 1|23, 2|13 \text{ and } 1|2|3$.

9. (20 points) Let $N = 2^2 \cdot 3^3 \cdot 5^5$ and $M = 2^4 \cdot 3 \cdot 5^5 \cdot 7^3$.

(a) What is the greatest common divisor of $M$ and $N$?
Solution: $gcd = 2^2 \cdot 3 \cdot 5 = 60$.

(b) What is the least common multiple of $M$ and $N$?
Solution: $lcm = 2^4 \cdot 3^3 \cdot 5^5 \cdot 7^3 = 463050000$

(c) How many (positive integer) divisors does $N$ have?
Solution: The number of divisors of $N$ is $(2 + 1)(3 + 1)(5 + 1) = 72$

(d) How many of the numbers $1, 2, 3, 4, \ldots, 1000$ have an odd number of divisors?
Solution: A number has an odd number of divisors precisely when it is a perfect square. There are \lfloor \sqrt{1000} \rfloor = 31 \text{ perfect squares less than } 1000.$
10. (16 points) Let $A = \{1, 2, 3, 4\}$. Find examples of relations on $A$ which satisfy each of the following collections of conditions:

(a) $R$ is reflexive, is symmetric, is not transitive and $|R| = 8$.

**Solution:** $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (2, 3), (3, 2)\}$ is both reflexive and symmetric but not transitive.

(b) $R$ is symmetric, not transitive and $|R| = 2$.

**Solution:** $R = \{(1, 2), (2, 1)\}$ is symmetric but not transitive.

(c) $R$ is symmetric and antisymmetric and $|R| = 3$.

**Solution:** $R = \{(1, 1), (2, 2), (3, 3)\}$ is both symmetric and antisymmetric.

(d) Symmetric and transitive and $|R| = 5$.

**Solution:** $R = \{(1, 2), (2, 1), (1, 2), (2, 2), (3, 3)\}$ is symmetric and transitive.