March 26, 2004       Name
There are 135 points available on this test. Each question is marked with its value. To get full credit for a problem, you must show your work. Correct answers with incorrect supporting work will receive substantially reduced credit.

1. (15 points) Let \( p(x) = x^2 - 4x + 5 \).
   (a) Compute \( p'(x) \)
   \[ \text{Solution: } p'(x) = 2x - 4. \]
   (b) Compute \( p''(x) \)
   \[ \text{Solution: } p''(x) = 2 \]
   (c) Use the information in (a) to find an equation for the line tangent to the graph of \( p \) at the point \((1, 2)\).
   \[ \text{Solution: } y - 2 = p'(1)(x - 1) = -2(x - 1), \text{ so } y = -2x + 4. \]

2. (20 points) Consider the astroid \( x^{2/3} + y^{2/3} = 4. \)
   (a) Show that the point \((-3\sqrt{3}, 1)\) belongs to the graph.
   \[ \text{Solution: } (-3\sqrt{3})^{2/3} + 1^{2/3} = (9 \cdot 3)^{1/3} + 1 = 4. \]
   (b) Find \( y' \) as a function of \( x \) and \( y \) using implicit differentiation.
   \[ \text{Solution: } \text{Differentiate both sides to get } \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{1/3} \cdot y' = 0, \text{ so } y' = \frac{-2x^{-1/3}}{\frac{2}{3}y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}. \]
   (c) Find the slope of the line tangent to the curve at the point \((-3\sqrt{3}, 1)\).
   \[ \text{Solution: } m = -\left(\frac{1}{-3\sqrt{3}}\right)^{1/3} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \approx 0.5773. \]
   (d) Find an equation for the tangent line whose slope you found above.
   \[ \text{Solution: } \text{Use the point-slope form to get } y - 1 = \frac{\sqrt{3}}{3}(x + 3\sqrt{3}) = \frac{\sqrt{3}}{3}x + 3. \]
   Thus, \( y = \frac{\sqrt{3}}{3}x + 4 \)
3. (30 points) Suppose the functions $f$ and $g$ are given partially by the table of values shown. The next problems refer to the functions $f$ and $g$ given in the tables. Consider the table of values given for the functions $f, f', g,$ and $g'$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$f'(x)$</th>
<th>$g(x)$</th>
<th>$g'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>3</td>
<td>2</td>
<td>5</td>
<td>2</td>
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<tr>
<td>1</td>
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<td>6</td>
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<td>3</td>
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</tbody>
</table>

(a) Let $K(x) = f \circ g(x)$. Compute $K'(3)$

**Solution:**

$$K'(3) = f'(g(3)) \cdot g'(3) = f'(3) \cdot g'(3) = 16.$$  

(b) Let $L(x) = f(x) \cdot g(x)$. Compute $L'(2)$.

**Solution:**

$$L'(2) = f'(2)g(2) + g'(2)f(2) = 17.$$  

(c) Let $U(x) = f \circ f(x)$. Compute $U'(1)$.

**Solution:**

$$U'(1) = f'(f(1)) \cdot f'(1) = f'(3) \cdot f'(1) = 4 \cdot 5 = 20.$$  

(d) Let $V(x) = g(x)/f(x)$. Compute $V'(4)$.

**Solution:**

$$V'(4) = \frac{g'(4)f(4) - f'(4)g(4)}{f(4)^2} = \frac{5 \cdot 4 - 6 \cdot 1}{16} = \frac{14}{16} = \frac{7}{8}.$$  

(e) Let $W(x) = (g(x))^2$. Compute $W'(5)$.

**Solution:**

$$W'(5) = 2g(5)g'(5) = 2 \cdot 2 \cdot 4 = 16.$$  

(f) Let $Z(x) = g(x^2 \cdot f(x))$. Compute $Z'(1)$.

**Solution:**

$$Z'(1) = g'(1^2f(1))(2 \cdot 1f(1) + f'(1) \cdot 1^2) = g'(3)(2 \cdot 3 + 5) = 4 \cdot 11 = 44.$$
4. (25 points)

(a) Find \( \frac{d}{dx}(\sin x) \)

Solution: \( \frac{d}{dx}(\sin x) = \cos x \)

(b) Write an equation involving the functions \( \sin \) and \( \sin^{-1} \), the composition operation, and the identity function. In other words write an equation that shows you know what \( \sin^{-1} x \) is.

Solution: \( \sin \circ \sin^{-1}(x) = x \) or \( \sin(\sin^{-1}(x)) = x \).

(c) Differentiate both sides of the equation in (b).

Solution: Let \( y = \sin^{-1}(x) \). By the chain rule, \( \cos(y) \cdot y' = 1 \), so \( y' = 1/\cos y = 1/\cos(\sin^{-1}(x)) \).

(d) Use the result in (c) to find an expression for \( \frac{d}{dx}(\sin^{-1} x) \).

Solution: Using the triangle with sides \( x, \sqrt{1-x^2}, \) and 1, it follows that \( \frac{d}{dx}(\sin^{-1} x) = 1/\sqrt{1-x^2} \).

(e) Let \( h(x) = \sin^{-1}(x^2) \). Compute \( h'(x) \).

Solution: Using the chain rule, \( h'(x) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}} \).
5. (25 points) Compute the following derivatives.

(a) \( \frac{d}{dx} e^{\sin x} \)

\textbf{Solution:} \( \frac{d}{dx} e^{\sin x} = e^{\sin x} \cdot \cos x \)

(b) \( \frac{d}{dx} \ln(\tan x) \)

\textbf{Solution:} \( \frac{d}{dx} \ln(\tan x) = \frac{1}{\tan x} \cdot \sec^2 x = \csc x \cdot \sec x. \)

(c) \( \frac{d}{dx} \sqrt{x} \ln x \)

\textbf{Solution:} \( \frac{d}{dx} \sqrt{x} \ln x = \frac{1}{2} x^{-1/2} \ln x + \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} \left( \frac{\ln x}{2} + 1 \right). \)

(d) \( \frac{d}{dx} (\cos(x^2))^3 \)

\textbf{Solution:} This is a triple composition, so you use the chain rule twice: \( \frac{d}{dx} (\cos(x^2))^3 = -3(\cos(x^2))^2 \cdot \sin x^2 \cdot 2x. \)

(e) \( \frac{d}{dx} \tan^{-1}(2x) \)

\textbf{Solution:} \( \frac{d}{dx} \tan^{-1}(2x) = \frac{1}{1+(2x)^2} \cdot 2 = \frac{2}{1+4x^2}. \)
6. (20 points) Suppose $f$ is defined by:

$$f(x) = \begin{cases} 
\ln(3x) & \text{if } x > 0 \\
\ln(-x) & \text{if } x < 0 
\end{cases}$$

(a) Find $f'(3)$.

**Solution:** Near $x = 3$, $f'(x) = \frac{1}{3x} \cdot 3$, so $f'(3) = 1/3$.

(b) Find $f'(-e)$.

**Solution:** Near $-e$, $f'(x) = (1/x) \cdot -1 = 1/x$, so $f'(-e) = -1/e$.

(c) Find an equation for the line tangent to the graph of $f$ at the point $(-e, f(-e))$.

**Solution:** Since $m = -1/e$ and $f(-e) = \ln(-(-e)) = 1$, it follows that an equation for the line is $y - 1 = -\frac{x}{e} - 1$, or $y = -\frac{x}{e}$.