The Theory of Linear Homogeneous (Constant Coefficient) Recurrences

by Evan Houston

The paper below was prepared by Evan Houston for his Combinatorics classes several years ago. The mathematics is due to him. The typo’s are all mine (Harold Reiter).

1. Vector spaces. Recall that a vector space (over $\mathbb{R}$) is a set $V$, together with operations of vector addition and scalar (real number) multiplication, which satisfy a long list of axioms. The most familiar examples are the spaces $\mathbb{R}^n$ of $n$-tuples of real numbers with addition and scalar multiplication defined by $(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$ and $k \cdot (a_1, \ldots, a_n) = (ka_1, \ldots, ka_n)$ However, equally important are function spaces: for any set $A$ we consider the set $V$ of all functions $f: \mathbb{N} \to \mathbb{R}$, with addition and scalar multiplication defined by $f(a) + g(a) = f(a) + g(a)$ and $(kf)(a) = k(f(a))$. We shall see that the set of solutions to a linear homogeneous recurrence relation is a subspace of the vector space of all functions $f: \mathbb{N} \to \mathbb{R}$, where $\mathbb{N}$ is the set of all nonnegative integers.

2. The general linear homogeneous recurrence with constant coefficients is given by

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ra_{n-r}. \quad (*)$$

By a solution to (*) we mean a function $f: \mathbb{N} \to \mathbb{R}$ for which

$$f(n) = c_1f(n-1) + c_2f(n-2) + \cdots + c_rf(n-r) \text{ for } n \geq r.$$  

Of course, all we have done is replace a subscript by a variable, but this emphasizes the fact that solutions to (*) are functions.

3. Theorem. The set of solutions to (*) is a subspace of the space of all functions from $\mathbb{N}$ into $\mathbb{R}$. That is, if $f$ and $g$ are solutions, then so is $kf + lg$, for all scalars $k$ and $l$.

Proof: We are given that $f(n) = c_1f(n-1) + c_2f(n-2) + \cdots + c_rf(n-r)$ and that $g(n) = c_1g(n-1) + c_2g(n-2) + \cdots + c_rg(n-r)$ for $f: \mathbb{N} \to \mathbb{R}$. Thus

$$(kf + lg)(n) = kf(n) + kg(n)$$
$$= c_1[kf(n-1) + lg(n-1)] + \cdots + c_r[kf(n-r) + lg(n-r)]$$
$$= c_1[(kf + lg)(n-1)] + \cdots + c_r[(kf + lg)(n-r)].$$

4. The characteristic equation of the linear homogeneous recurrence (*) is defined to be $\alpha^n - c_1\alpha^{n-1} - \cdots - c_r\alpha^{n-r} = 0$. If we ignore the root $\alpha = 0$ to the characteristic equation, we can rewrite it as $\alpha^r - c_1\alpha^{r-1} - \cdots - c_{r-1}\alpha - c_r = 0$. 

1
5. **Theorem.** If $\alpha_1$ is a root of the characteristic equation of the homogeneous linear recurrence $(*)$, then $f(n) = \alpha_1^n$ is a solution to $(*)$. It follows that if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are roots of $(*)$, then so is $A_1\alpha_1^n + A_2\alpha_2^n + \ldots + A_k\alpha_k^n$, for any constants $A_1, A_2, \ldots, A_k$.

**Proof:** The "It follows" statement follows from the first statement by (3) above. Because $\alpha_1$ is a solution to the characteristic equation, it follows that $f(n) = \alpha_1^n$, which, in turn equal to $c_1\alpha_1^{n-1} + \cdots + c_r\alpha_1^{n-r} = c_1f(n-1) + \cdots + c_rf(n-r)$, as desired.

6. Brief discussion of what we want to do. We wish to describe the general solution to $(*)$. The main complication is that the theory splits into two cases, according to whether the characteristic equation has distinct or repeated roots. However, in each case it is possible to describe the general solution. This amounts to giving a basis for the space of solutions (see (3) above). Recall that a basis is a linearly independent spanning set for the space; basically (no pun intended), a basis is a set $\{f_1, f_2, \ldots, f_r\}$ of solutions such that every solution can be written as a linear combination of the functions in the set (that is, it spans), and such that the set is minimal with this property (linearly independent). We shall begin with the case of distinct roots. (We may as well be truthful: we shall only indicate how to handle the case of repeated roots.)

7. **Theorem.** Let $M$ be an $n \times n$ matrix with real entries. Then the following statements are equivalent.

(a) $M$ is invertible.

(b) The homogeneous linear system $MX = 0$ has only the trivial solution $X = 0$.

(c) The linear system $MX = B$ has a solution for each $B$. Here, $B$ stands for the right-hand side of the system; that is, $B$ is a column of $n$ real numbers.

**Proof:** See your linear algebra text.

8. **Lemma.** (Actually, this is the most important part of the main theorem below.) Assume that the characteristic equation to the linear homogeneous recurrence $(*)$ has $r$ distinct roots $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$. If $f$ is a solution to $(*)$, then $f(n) = A_1\alpha_1^n + A_2\alpha_2^n + \ldots + A_k\alpha_k^n$, for some constants $A_1, A_2, \ldots, A_k$.

**Proof:** Since $f$ is a solution to $(*)$, we have $f(n) = c_1f(n-1) + \cdots + c_rf(n-r)$. Note that $f$ is therefore completely determined as soon as we know the values $f(0), f(1), \ldots, f(r-1)$. To put another way, it suffices to show that we can find constants $A_1, A_2, \ldots, A_r$ such that $f(i) = A_1\alpha_1^i + A_2\alpha_2^i + \ldots + A_r\alpha_r^i$ for each
This amounts to showing that the linear system \( MX = B \) has a solution, where

\[
M = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_r \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{r-1} & \alpha_{r-1} & \cdots & \alpha_r^{-1}
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
f(0) \\
f(1) \\
\vdots \\
f(r-1)
\end{bmatrix}
\]

By (7) above it suffices to show that the matrix \( M \) is invertible. Now it is well known that \( M \) is invertible if, and only if, its transpose is invertible. Thus we need only show that \( M^tX = 0 \) has only the trivial solution, again by (7). Accordingly, suppose that the

\[
X_0 = \begin{bmatrix}
d_0 \\
d_1 \\
\vdots \\
d_{r-1}
\end{bmatrix}
\]

is a solution to \( M^tX = 0 \). This gives \( d_0 + d_1\alpha_i + \cdots + d_{r-1}\alpha_i^{r-1} = 0 \) for each \( i = 0,1,\ldots,r \). That is, each of \( \alpha_1, \alpha_2,\ldots,\alpha_r \) is a root of the polynomial \( d_0 + d_1x + \cdots + d_{r-1}x^{r-1} \). However, a nonzero polynomial of degree \( r-1 \) can have at most \( r-1 \) roots, whence this must be the zero polynomial. Therefore, each of \( d_0, d_1,\ldots,d_r-1 \) must be zero, as was to be shown. This completes the proof.

9. **Lemma.** As before, assume that the roots of the characteristic equation are the distinct numbers \( \alpha_1, \alpha_2,\ldots,\alpha_r \). Let \( f_i = \alpha_i^n \). Then the \( f_i \) are linearly independent; that is, the only solution to \( e_1f_1 + \cdots + e_rf_r = 0 \) is \( e_1 = e_2 = \cdots e_r = 0 \). Proof: If we plug in the values \( n = 0,1,\ldots,r-1 \) into \( e_1f_1 + \cdots + e_rf_r = 0 \), we get that \( e_1\alpha_i^1 + e_2\alpha_i^2 + \cdots + e_r\alpha_i^r = 0 \) for each \( i = 0,1,\ldots,r-1 \). This means that \( ME = 0 \), where \( M \) is the matrix above and \( E \) is the column matrix whose entries are the \( e_j \). Since \( M \) is invertible, we must have all \( e_j \) equal to zero.

10. **Theorem.** Assume that the characteristic equation to (*) has distinct roots \( \alpha_1, \alpha_2,\ldots,\alpha_r \). Then the functions \( f_1(n) = \alpha_1^n, \quad f_2(n) = \alpha_2^n,\ldots,f_r(n) = \alpha_r^n \), form a basis for the space of all solutions to (*). Proof: This now follows immediately from the lemmas above. We will now turn to a discussion of the case in which the characteristic equation has repeated roots.

11. **Repeated roots.** We say that \( \alpha_1 \) is a root of multiplicity \( k \) of the polynomial \( h(\alpha) = d_0 + d_1\alpha + \cdots + d_m\alpha^m = 0 \) if the polynomial can be factored in the form \( h(\alpha) = d_0 + d_1\alpha + \cdots + d_m\alpha^m = 0 = (\alpha - \alpha_1)^k g(\alpha) \), where \( g(\alpha_1) \neq 0 \).
For example, $\alpha = 3$ is a root of multiplicity 2 of the polynomial $(\alpha^3)^2(\alpha + 4) = \alpha^3 - 2\alpha^2 - 15\alpha + 36$.

12. Example. Let $H(\alpha) = (\alpha^3)^2(\alpha + 4) = \alpha^3 - 2\alpha^2 - 15\alpha + 36$. Note that the derivative $H'$ satisfies $H'(\alpha) = 3\alpha^2 - 4\alpha - 15 = (\alpha - 3)(3\alpha + 5)$ and that $\alpha = 3$ is a root of multiplicity 1 of the polynomial $H'(\alpha)$. This is no accident.

13. Lemma. Let $h(\alpha) = d_0 + d_1\alpha + \cdots + d_n\alpha^n$. If $\alpha_1$ is a root of multiplicity $k$ of $h(\alpha)$, then $\alpha_1$ is a root of multiplicity $k - 1$ of the polynomial $h'(\alpha)$, where $h'(\alpha)$ is the derivative of $h$.

Proof: If $h(\alpha) = (\alpha - \alpha_1)^kg(\alpha)$, with $g(\alpha) \neq 0$, then $h'(\alpha) = (\alpha - \alpha_1)^{k-1}g(\alpha) + (\alpha - \alpha_1)g'(\alpha) = (\alpha - \alpha_1)^{k-1}[kg(\alpha) + (\alpha - \alpha_1)g'(\alpha)]$. Since $(\alpha = \alpha_1)$ is not a root of $g(\alpha)$, it is not a root of the bracketed expression either. Hence $h(\alpha) = \alpha_1$ is a root of multiplicity $k-1$ of the polynomial $g'(\alpha)$.

14. Theorem. If $\alpha$ is a root of multiplicity $k$ of the characteristic equation of $(*)$, then $f_0(n) = \alpha_n^n$, $f_1(n) = n\alpha_1^n$, $f_2(n) = n^2\alpha_1^n$, $f_{k-1}(n) = n^{k-1}\alpha_1^n$ are all solutions to $(*)$.

Proof: Certainly, $f_0$ is a solution of $(*)$. Now let $h(\alpha)$ be the characteristic equation. By (13), $\alpha_1$ is a root of multiplicity $k - 1$ of $h'(\alpha) = n\alpha^n - c_1(n - 1)\alpha^{n-1} - \cdots - c_r(n - r)\alpha^{n-r-1}$. It is harmless to multiply through by $\alpha_1$ yielding $n\alpha_1 = c_1(n - 1)\alpha^{n-1} - \cdots - c_r(n - r)\alpha^{n-r}$, which shows that $f_1$ is also a solution of $(*)$. Let $h_1(\alpha) = (\alpha)h'(\alpha)$. We have seen that $\alpha_1$ is a root of multiplicity $k - 1$ of $h_1(\alpha)$. Applying (13) again and multiplying through by $\alpha$ yields that $\alpha_1$ is a root of multiplicity $k - 2$ of $h_2(\alpha) = (\alpha)h'(\alpha)$, whence $n^2\alpha_1^n = c_1(n - 1)^2\alpha_1^{n-1} + \cdots + c_r(n - r)^2\alpha^{n-r}$, and $f_2$ is a solution. If we continue in this manner, we can see that each $f_i$ is a solution of $(*)$.

15. Discussion and general giving up. Using the notation and hypothesis of (14), it follows from (3) that $A_0f_0 + A_1f_1 + \cdots + A_{k-1}f_{k-1}$ is also a solution to $(*)$ for each choice of constants $A_0, A_1, \cdots, A_{k-1}$. Now the idea is to combine these solutions for each root $\alpha_j$ to get the general solution. The whole thing boils down to the invertibility of a certain matrix, just as in the case of distinct roots. However, it is not quite so easy to demonstrate invertibility of the matrix that arises in this case, so we omit the details. Here is the general theorem.

16. Let $\alpha_1, \alpha_2, \ldots, \alpha_p$ be all the roots of the characteristic equation of $(*)$, and assume that $\alpha_i \neq \alpha_j$ for $i \neq j$. Further assume that $\alpha_i$ is a root of multiplicity $k_i$. For each $i$, let $f_{ij}(n) = n^j\alpha_i^n$, for $j = 0, 1, \ldots, k_i - 1$. Then the functions $f_{ij}$ form a basis for the space of all solutions to $(*)$. In particular, the general
solution to (*) is given by

$$\sum_{i=1}^{p} \sum_{j=0}^{k_i} A_{ij} f_{ij}.$$ 

17. **Remark.** Note that (16) subsumes (10), since in (16) we can take each $k_i = 1$. 