

UNC Charlotte 2012 Comprehensive

March 5, 2012

1. In the English alphabet of capital letters, there are 15 “stick” letters which contain no curved lines, and 11 “round” letters which contain at least some curved segment. How many different 3-letter sequences can be made of two different stick letters and one curved letter?

Stick: A E F H I K L M N T V W X Y Z

Round: B C D G J O P Q R S U

(A) 2310 (B) 4620 (C) 6930 (D) 13860 (E) none of these

Solution: (C). 6930. Without regard to order, there are $C(15, 2) = 105$ choices for the two stick letters and 11 choices for the round letter. Allowing for different orders, there are $3! = 6$ ways to order the chosen letters, so the total number with order considered is $6 \times 105 \times 11 = 6930$.

2. I have several quarters. If I divide my quarters into two unequal piles, then the difference of the squares of the number of coins in each pile is 24 times the difference of the number of coins in each pile. How much is my collection of quarters worth?

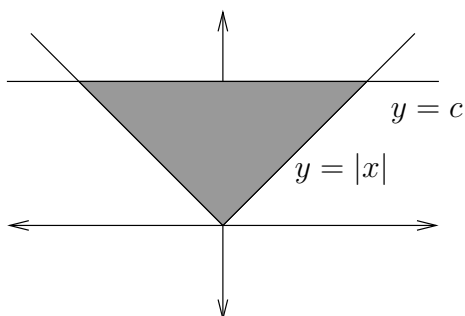
(A) \$9 (B) \$8 (C) \$7 (D) \$6 (E) \$5

Solution: (D). Let x be the number of coins in the larger pile and y be the number of coins in the smaller pile. Then $x^2 - y^2 = 24(x - y)$. Dividing both sides by $x - y$ (since $x - y$ is nonzero) yields $x + y = 24$. So I have \$6 worth of quarters.

3. The area bounded by the graph of the function $y = |x|$ and by the line $y = c$ is 5. What is c equal to?

(A) $\sqrt{5}$ (B) $2\sqrt{5}$ (C) 5 (D) 10 (E) 25

Solution: (A). The area bounded by the two graphs is a triangle with base $2c$ and height c . Thus its area is $c^2 = 5$ hence $c = \sqrt{5}$.



4. In the Carolina Cash 5 game, you have to select five different numbers from 1 to 39. You win some prize if two or more of your five chosen numbers match the five numbers drawn (order does not matter). What is the probability that you win some prize in a single drawing? Round your answer to the nearest percent.
- (A) (7%) (B) (8%) (C) (9%) (D) (11%) (E) (12%)

Solution: (D). There are $\binom{39}{5}$ ways to draw five numbers altogether. You do not win if the set of five numbers drawn is disjoint of our selection or matches exactly one number. There are $\binom{34}{5}$ ways to select a disjoint set and $5\binom{34}{4}$ ways to draw five numbers that match exactly one of our numbers. Here 5 is the number of ways to select the matching number, and $\binom{34}{4}$ is the number of ways to select four numbers that do not match our selection. The probability that we do not win is

$$\frac{\binom{34}{5} + 5\binom{34}{4}}{\binom{39}{5}} = \frac{278,256 + 231,880}{575,757} \approx 0.89.$$

Thus the probability of winning is $1 - 0.89 = 0.11$, that is, 11%.

5. Let a and b be the two roots of the equation $x^2 + 3x - 3 = 0$. Evaluate the value $a^2 + b^2$.
- (A) 4 (B) 9 (C) 10 (D) 12 (E) 15

Solution: (E). Since $a + b = -3$ and $ab = -3$ so we have $a^2 + b^2 = (a + b)^2 - 2ab = (-3)^2 - 2 \cdot (-3) = 15$.

6. One sphere is inscribed in a cube, while the cube is also inscribed in another sphere. Find the ratio of the volumes of the larger sphere to the smaller sphere.

(A) $\sqrt{3}$ (B) $2\sqrt{3}$ (C) $3\sqrt{3}$ (D) $3\sqrt{2}$ (E) none of these

Solution: (C). The inscribed sphere has a radius r and $V = (4/3)\pi r^3$, while the circumscribed sphere has the radius which goes to the vertex of the cube. Clearly the inscribed radius r is half the edge of the cube, so that one cube vertex has coordinate (r, r, r) if the center of the spheres were the origin. The circumscribed radius is thus $\sqrt{r^2 + r^2 + r^2} = r\sqrt{3}$. The circumscribed volume is then $V = (4/3)\pi(r\sqrt{3})^3 = (4/3)\pi(3\sqrt{3})r^3$. The ratio is thus $3\sqrt{3}$.

7. Let ABC be an equilateral triangle with an inscribed circle of radius 1, find the length of AB.

(A) $\sqrt{2}$ (B) $2\sqrt{2}$ (C) $\sqrt{3}$ (D) $2\sqrt{3}$ (E) $3\sqrt{3}$

Solution: (D). Let O be the center of the circle and M be the intersection point of the circle with AB . Then $\angle OAM = 30^\circ$ with $OM = 1$ hence $OA = 2$. Thus $OM = \sqrt{3}$ and $AB = 2OM = 2\sqrt{3}$.

8. Determine the sum of the y -coordinates of the three intersection points of $y = x^2 - x - 5$ and $y = -5/x$.

(A) -5 (B) -3 (C) 0 (D) 3 (E) 5

Solution: (A). Setting both equations equal to each other you get $x = 1$, $x = \pm\sqrt{5}$. It follows that the corresponding y -coordinates are -5 and $\pm\sqrt{5}$ so the sum of the three y -coordinates is -5 .

9. Consider a five-digit integer of the form $a11bc$ (with $a \geq 1$). It is known that it is divisible by 45. How many such numbers exist?

(A) 21 (B) 20 (C) 19 (D) 18 (E) 10.

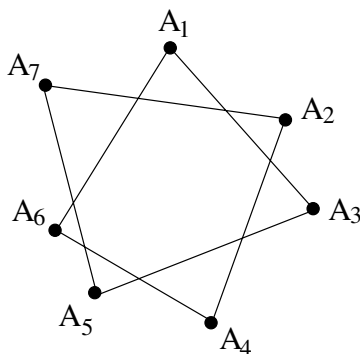
Solution: (B). c can be either 0 or 5 and $a + b + c + 2$ is divisible by 9. Two cases:

a) $c = 0$ and $a + b + 2$ is divisible by 9, so $a + b = 16$ (3 cases: $(7, 9)$, $(8, 8)$, $(9, 7)$) or $a + b = 7$ (7 cases: $(1, 6)$, $(2, 5)$, ..., $(7, 0)$). b) $c = 5$ and $a + b + 7$ is divisible by 9, so $a + b = 11$ (8 cases: $(2, 9)$, $(3, 8)$, ..., $(9, 2)$) or $a + b = 2$ (2 cases: $(2, 0)$ and $(1, 1)$). Thus the answer is $3 + 7 + 8 + 2 = 20$.

10. Consider a general convex 7-gon with vertices A_1, A_2, \dots, A_7 (marked in the order as they appear on the polygon). Connecting the first vertex A_1 with the third vertex A_3 , the second vertex A_2 with the fourth vertex A_4 , and so on,

we get a “star” as shown in the figure below with the vertices A_1, A_2, \dots, A_7 . Find the sum of the angles $\angle A_1$ to $\angle A_7$.

- (A) 360° (B) 480° (C) 540° (D) 600° (E) 620°



Solution: (C). For a triangle $\triangle XYZ$, the sum of the angles with vertices at X and Y is equal to the exterior angle at Z . Since $2\angle X + \angle Y + \angle Z = \angle X + 180^\circ$, the sum of the exterior angles at Y and Z is equal to 180° plus $\angle X$. Thus the sum of the angles at A_1 through A_7 is twice the angle sum of the interior heptagon minus $7 \cdot 180^\circ$. For the heptagon the angle sum is $7 \cdot 180^\circ - 360^\circ = 5 \cdot 180^\circ$. So the desired sum is $10 \cdot 180^\circ - 7 \cdot 180^\circ = 540^\circ$.

11. After a ship wreck, a surviving mouse finds himself on an uninhabited island with one kilogram of cheese and no other food. On the 1st day, he eats $1/4$ of the cheese, on the 2nd, $1/9$ of the remaining cheese, on the 3rd, $1/16$ of the remainder, and so on. Let M be the total amount of cheese (measured in kilograms) consumed during the first 6 weeks. Which of the following statements is true?

- (A) $0.4878 < M \leq 0.4881$ (B) $0.4881 < M \leq 0.4884$
 (C) $0.4884 < M \leq 0.4887$ (D) $0.4887 < M \leq 0.4890$
 (E) $0.4890 < M \leq 0.4893$

Solution: (B). The initial amount of cheese is 1kg; since on the k th day the mouse eats $\frac{1}{(k+1)^2}$ of the remaining cheese, the amount remaining after n days is

$$\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \cdots \left(1 - \frac{1}{(n-1)^2}\right) \left(1 - \frac{1}{n^2}\right) \left(1 - \frac{1}{(n+1)^2}\right)$$

$$\begin{aligned}
&= \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdots \frac{(n-2) \cdot n}{(n-1) \cdot (n-1)} \cdot \frac{(n-1) \cdot (n+1)}{n \cdot n} \cdot \frac{n \cdot (n+2)}{(n+1) \cdot (n+1)} \\
&= \frac{1}{2} \cdot \frac{n+2}{n+1},
\end{aligned}$$

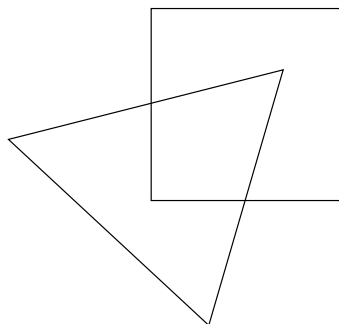
so that the consumed amount is

$$1 - \frac{1}{2} \cdot \frac{n+2}{n+1} = \frac{1}{2} \cdot \frac{n}{n+1}.$$

For $n = 42$, this is

$$\frac{1}{2} \cdot \frac{42}{43} \approx 0.48837.$$

12. Two adjacent sides of the unit square and two sides of an equilateral triangle bisect each other. Find the area of the equilateral triangle.
 (A) $\sqrt{3}/2$ (B) 1 (C) 0.8 (D) $\sqrt{2}$ (E) $\sqrt{2}/2$



Solution: (A). Let M and N be the two points where the two sides of the unit square and two sides of an equilateral triangle bisect each other. Their distance from each other is $|MN| = \frac{\sqrt{2}}{2}$. Hence the side of the equilateral triangle is $a = \sqrt{2}$. Its area is $\frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{2}$.

13. Let C be a circle that intersects each of the circles $(x+2)^2 + y^2 = 2^2$, $(x-4)^2 + (y-2)^2 = 2^2$ and $(x-4)^2 + (y+2)^2 = 2^2$ in exactly one point and does not contain any of these circles inside it. If the radius r of C is of the form $r = p/q$ where p and q are relatively prime integers, what is $p+q$?
 (A) 5 (B) 7 (C) 9 (D) 11 (E) 13

Solution: (B). The center of C is of the form $(r, 0)$ by symmetry. Finding its distances to the center $(4, 2)$ of $(x - 4)^2 + (y - 2)^2 = 2^2$ and to the center $(-2, 0)$ of $(x + 2)^2 + y^2 = 2^2$ lead to $(r + 2)^2 = (r - 4)^2 + (0 - 2)^2$. Solving for r , we get $r = \frac{4}{3}$.

14. Three crazy painters painted the floor in three different colors. One painted 75% of the floor in red on Monday, the second painted 70% of the floor in green on Tuesday, and the third painted 65% of the floor in blue on Wednesday. At least what percent of the floor must be painted in all the three colors?

(A) 10% (B) 25% (C) 30% (D) 35% (E) 65%

Solution: (A). 25% of the floor is NOT painted in red. 30% of the floor is NOT painted in green. 35% of the floor is NOT painted in blue. $25\% + 30\% + 35\% = 90\%$ That means at least 10% are not painted in the three color. One can show that such coloring exists.

15. After each mile on a highway from Charlotte (CLT) to the City of Mathematics Students (CMS) there is a two sided distance marker. One side of the marker shows the distance from CLT while the opposite side shows the distance to CMS. A curious student noticed that the sum of the digits from the two sides of each marker stays a constant 13. Find the distance d between the cities.

(A) $10 \leq d < 20$ (B) $20 \leq d < 30$ (C) $30 \leq d < 40$ (D) $40 \leq d < 50$
(E) $50 \leq d < 60$

Solution: (D). The distance cannot exceed 49 miles since otherwise the marker at 49 miles will have 49 on one side and a positive number on the other side and the sum of the digits would be greater than 13. Now look at the markers at one mile and ten miles. On the other side of these markers, we must have one of the numbers 48 and 39. Thus the first one must be 48 and the second one 39. Thus, the distance can only be 49 miles. On the other hand, one can check that the distance 49 does give a solution.

16. Consider the sum $1^2 + 2^2 + \dots + 2012^2$. What is its last digit?

(A) 6 (B) 5 (C) 3 (D) 1 (E) 0

Solution: (E). Using the formula $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$, we have $1^2 + 2^2 + \dots + 2012^2 = \frac{2012 \cdot 2013 \cdot 4025}{6} = 1006 \cdot 671 \cdot 4025 \equiv 0 \pmod{10}$. So the last digit is 0. Alternatively, using mod(10) relation, $1^2 + 2^2 + 3^2 + \dots + 10^2 \equiv 5 \pmod{10}$ and this repeats for every 10 consecutive integers, $2012 = 201 \times 10 + 2$,

$2001^2 \equiv 1 \pmod{10}$ and $2012^2 \equiv 4 \pmod{10}$ so $1^2 + 2^2 + \cdots + 2012^2 \equiv 201 \times 5 + 1 + 4 \equiv 0 \pmod{10}$.

17. Infinitely many empty boxes, each capable of holding six dots are lined up from right to left. Each minute a new dot appears in the rightmost box. Whenever six dots appear in the same box, they **fuse** together to form one dot in the next box to the left. How many dots are there after 2012 minutes? For example, after seven minutes we have just two dots, one in the rightmost box and one in the next box over.

(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

Solution: (E). This is just base 6 enumeration in disguise. The base 6 representation of 2012 is 13152, so there are $1 + 3 + 1 + 5 + 2 = 12$ dots in the boxes after 2012 minutes.

18. Consider the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21 \dots$ where each term, after the first two, is the sum of the two previous terms. How many of the first 1000 terms are divisible by 3?

(A) 200 (B) 250 (C) 299 (D) 300 (E) 333

Solution: (B). Consider the sequence in terms of its remainders when divided by 3, one gets a repeating cycle of length 8: $1, 1, 2, 0, 2, 2, 1, 0, \dots$. That is, every fourth of the numbers in the sequence is divisible by 3. Thus the answer is $1000/4 = 250$.

19. The points A, B, C and D are the vertices of a unit square. How many squares (including $ABCD$ itself) in the same plane have two or more of these points as vertices?

(A) 13 (B) 12 (C) 9 (D) 5 (E) 4

Solution: (A). There are squares of three sizes. There are 4 of area $1/2$, 5 of area 1 and 4 of area 2.

20. Three fair (six sided) dice are colored green, red and yellow respectively. Each die is rolled once. Let g, r and y be the resulting values of the green, red and yellow dice respectively. What is the probability that $g \leq r \leq y$?

(A) $\frac{5}{27}$ (B) $\frac{6}{27}$ (C) $\frac{7}{27}$ (D) $\frac{8}{27}$ (E) $\frac{5}{18}$

Solution: (C). There are $\binom{8}{3} = 56$ successful outcomes, so the probability is $\frac{56}{6^3} = \frac{7}{27}$.

21. For how many real numbers x is it true that $\lfloor \frac{x}{2} \rfloor + \lfloor \frac{2x}{3} \rfloor = x$ (where $\lfloor y \rfloor$ is the greatest integer n such that $n \leq y$)?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution: (D). x has to be an integer since the left side is an integer. It is easy to see that if $x \leq 1$ or $x \geq 6$ the equality does not hold. On the other hand 2, 3, 4 and 5 work.

22. A person is given a square-shaped pizza and a knife. He cuts the pizza with 40 straight cuts into as many pieces as possible. Denote the resulting number of pieces by N . Which of the following statements is true?
(A) $831 \leq N \leq 840$ (B) $821 \leq N \leq 830$ (C) $811 \leq N \leq 820$
(D) $801 \leq N \leq 810$ (E) $791 \leq N \leq 800$

Solution: (B). We first assume that the pizza occupies the entire plane. Denote the number of pieces after k cuts by N_k . We have $N_0 = 1$ and $N_1 = 2$. Furthermore, $N_2 = 4$, unless the 2nd cut is parallel to the 1st, in which case $N_2 < 4$. In general, $N_k = N_{k-1} + k$ if the k th cut intersects the first $k - 1$ cuts at $k - 1$ distinct points; otherwise, the difference $N_k - N_{k-1}$ will be smaller than k . Therefore, the largest possible value of N_k (which is attained if no cuts are parallel and no more than 2 cuts pass through a single point) is $1 + 1 + 2 + 3 + \dots + k = 1 + \frac{k(k+1)}{2}$. For $k = 40$, this is 821.

This result is true if the pizza occupies the entire plane. For a bounded square-shaped pizza, the number of pieces cannot be more than 821. (Indeed, any 40 cuts of the square can be continued to the plane, thus generating a partition of the plane into no more than 821 pieces, as we know; each of them, being intersected with the square, produces no more than one piece of the square. But this is the original partition of the square.) On the other hand, if we scale the above “maximal” partition of the plane so that each piece intersects the square, we get a partition of the square into 821 pieces.

23. Find the smallest five-digit integer such that the product of its digits is 2520. The sum of its digits is
(A) 25 (B) 26 (C) 27 (D) 29 (E) 30

Solution: (E). The smallest such integer is 15789 by looking at the prime factorization of 2250. So the answer is 30.

24. If a convex n -gon has no adjacent obtuse angles then n is at most

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Solution: (C). Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the angles of the n -gon. We distinguish two cases, depending on the parity of n . Assume first n is even. By the stated condition $\alpha_1 + \alpha_2 < 270^\circ$, $\alpha_3 + \alpha_4 < 270^\circ$, \dots , $\alpha_{n-1} + \alpha_n < 270^\circ$. Thus the sum of the angles is less than $n/2 \cdot 270^\circ = n \cdot 135^\circ$. On the other hand, the sum of the angles of a convex n -gon is $(n - 2) \cdot 180^\circ$. We obtain the inequality

$$(n - 2) \cdot 180 < n \cdot 135,$$

the solution of which is $n < 8$. Since n is even, this implies $n \leq 6$.

Assume n is odd. We may assume that α_n is not an obtuse angle, thus $\alpha_n \leq 90^\circ$. Similarly to the previous case, we have $\alpha_1 + \alpha_2 < 270^\circ$, $\alpha_3 + \alpha_4 < 270^\circ$, \dots , $\alpha_{n-2} + \alpha_{n-1} < 270^\circ$. Thus the sum of the angles is less than $(n - 1)/2 \cdot 270^\circ + 90^\circ = n \cdot 135^\circ - 45^\circ$. (The inequality is strict, since $n \geq 3$.) We obtain the inequality

$$(n - 2) \cdot 180 < n \cdot 135 - 45,$$

the solution of which is $n < 7$. Since n is odd, this gives $n \leq 5$.

Comparing the two cases we see that n can not exceed 6. There is a hexagon with the required properties, for example, we may choose $\alpha_1 = \alpha_3 = \alpha_5 = 90^\circ$ and $\alpha_2 = \alpha_4 = \alpha_6 = 150^\circ$.

25. You are in a large room with 50 ceiling lights (numbered from 1 to 50) that are changed from on to off or off to on by pulling a cord hanging from each light. Initially, all the lights are off. You begin by pulling the cord on every light (now they are all on). Then you pull the cord on light 2, 4, \dots , 50. After you finish that, you pull the cord on light 3, 6, \dots , 48. You repeat this with every fourth light, every fifth light, etc. until you pull the cord for every 50th light (only number 50, of course). How many lights are on at the end?

(A) 1 (B) 2 (C) 3 (D) 5 (E) 7

Solution: (E). Lights numbered 1, 4, 9, 16, 25, 36, and 49 are on. A cord must be pulled an odd number of times for the light to be on. Consider light number x and factor x as a product of primes. There are 15 primes (2, 3, 5, \dots , 47) that are less than 50. So $x = 2^{a_1} \cdot 3^{a_2} \cdot \dots \cdot 47^{a_{15}}$. There are $(a_1 + 1)(a_2 + 1)(a_{15} + 1)$ factors of this number. The cord is pulled once for each factor. Since we need an odd number of pulls for the light to be on, a_1, a_2, \dots, a_{15} must all be even, hence x is a perfect square.